ISRAEL JOURNAL OF MATHEMATICS 113 (1999), 125–162

DUAL PAIRS CORRESPONDENCES OF $E_{8,4}$ AND $E_{7,4}$

BY

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ABSTRACT

In this paper we state and prove the dual pairs correspondences of $\mathrm{Spin}(4,4) \times \mathrm{Spin}(8)$ in $E_{8,4}$ and $F_{4,4} \times \mathrm{SU}_2$ and $\mathrm{Spin}(4,4) \times \mathrm{SU}_2^3$ in $E_{7,4}$. For the first dual pair, every finite dimensional representation of Spin(8) occurs and each corresponds with finite but unbounded multiplicity to a quaternionic representation of Spin(4, 4) having the same infinitesimal character. For the other two dual pairs the correspondences are one to one.

1. Introduction

1.1 Let G_0 be the real adjoint Lie group of type $E_{8,4}$ or $E_{7,4}$ with real root system of type F_4. $E_{8,4}$ contains subgroup

$$\mathrm{Spin}(4,4) \times_{K_4} \mathrm{Spin}(8) \tag{1}$$

where $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the diagonal subgroup lying in the center of the spin subgroups. $E_{7,4}$ contains the subgroups

$$F_{4,4} \times \mathrm{SU}_2 \quad \text{and} \tag{2}$$

$$\mathrm{Spin}(4,4) \times_{K_4} (\mathrm{SU}_2(A') \times \mathrm{SU}_2(B') \times \mathrm{SU}_2(C')) \tag{3}$$

where $F_{4,4}$ is the simply connected split group. Here A', B', and C' are the indices to distinguish between the three SU_2's.

Received April 19, 1998

1.2 In [GW1] and [GW2], Gross and Wallach constructed the minimal representation π_Z of G_0 by continuation of the quaternionic discrete series. In this paper we will restrict the minimal representation to the three subgroups described above and derive their dual pair correspondences or Howe correspondences.

1.3 Before stating the main results, we need to define some notations. We denote the finite dimensional complex representation of a Lie group G or its Lie algebra $\mathfrak{g}$ with highest weight Λ by $\pi_G(\Lambda)$ or $\pi_{\mathfrak{g}}(\Lambda)$. Following [Boul] Planches, we will use $\varpi_1, \ldots, \varpi_n$ to denote the fundamental weights of G. Finally $S^n(\mathbb{C}^2)$ or S^n will denote the representation $\mathrm{Sym}^n(\mathbb{C}^2)$ of SU_2. μ_n will denote the finite cyclic group of order n. Suppose a finite group C is contained in both the centers of groups G_1 and G_2; then we denote

$$G_1 \times_C G_2 := (G_1 \times G_2)/\{(s,s)\colon s \in C\}.$$

1.4 Choose a positive root system of Spin(4,4) with respect to a compact Cartan subgroup such that the highest root $\tilde{\alpha}$ is a compact root. Denote the other three compact roots by A, B and C (see (53), (54) and (55)). Then Spin(4,4) has maximal compact subgroup

$$\mathrm{SU}_2(\tilde{\alpha}) \times_{\mu_2} (\mathrm{SU}_2(A) \times \mathrm{SU}_2(B) \times \mathrm{SU}_2(C)) \tag{4}$$

where each SU_2 corresponds to a compact root. Let $\Omega = n_1\varpi_1 + \cdots + n_4\varpi_4$ be a dominant weight of Spin(4,4) and let $\pi_{\rho(D_4)+\Omega}$ be the quaternionic discrete series representation with infinitesimal character $\rho(D_4) + \Omega$ and lowest K-type

$$S_{\tilde{\alpha}}^{n_1+2n_2+n_3+n_4+8} \otimes S_A^{n_1} \otimes S_B^{n_2} \otimes S_C^{n_2}.$$

It has Harish-Chandra module (see §3.5)

$$\mathbf{H}(\mathrm{Spin}(4,4), S_A^{n_1} \otimes S_B^{n_2} \otimes S_C^{n_2}[n_1 + 2n_2 + n_3 + n_4 + 10]).$$

We are ready to state the first main result of this paper:

THEOREM 1.4.1: *By restricting to the dual pair* $\mathrm{Spin}(4,4) \times_{K_4} \mathrm{Spin}(8)$, π_Z *decomposes as*

$$\mathrm{Res}^{E_{8,4}}_{\mathrm{Spin}(4,4)\times\mathrm{Spin}(8)} \pi_Z = \bigoplus_{\Omega = n_1\varpi_1 + \ldots + n_4\varpi_4} \Theta(\pi(\Omega)) \otimes \pi(\Omega) \tag{5}$$

where the sum is taken over all irreducible representations of Spin(8). $\Theta(\pi(\Omega))$ *is* (n_2+1) *copies of the quatenionic discrete series* $\pi_{\rho(D_4)+\Omega}$ *of* Spin(4,4).

1.5 There is a S_3 action on $E_{8,4}$ and when restricted to the maximal compact subgroup K becomes the S_3 outer automorphism (see §2.3 and [L1]). Observe that the above correspondence respects this automorphism.

1.6 The proof of the theorem occupies §5 to §8, which is almost independent of the rest of the paper. The method of proof is straightforward by first showing that $\Theta(\pi(\Omega))$ has the same K-types as those of the discrete series. The bulk of the proof is the branching rules given in §6 and §7. Finally we will use the branching rules, Blattner's formula and a result of Schmid and Hecht–Schmid given as Corollary 3.7.2 to conclude Theorem 1.4.1.

1.7 Since the Weyl group of D_4 contains -1, Spin(4, 4) and Spin(8) are inner forms of one another and hence they have the same Langlands L-group. The Langlands parameter of a discrete series is determined by its infinitesimal character. Therefore the correspondence in Theorem 1.4.1 agrees with the correspondence of the Langlands parameters induced by the identity map between the L-groups.

We would like to mention that B. Gross had previously conjectured that $\Theta(\pi(\Omega)) \supset \pi_{\rho(D_4)+\Omega}$.

1.8 The correspondences of the two dual pairs in $E_{7,4}$ are given as Theorems 12.2.1 and 12.4.1. They are formulated in terms of quaternionic representations which we will introduce in §3.

The proof of Theorems 12.2.1 and 12.4.1 are less direct as compared to Theorem 1.4.1 because in these cases we do not get the discrete series representations so neither Blattner formula nor Corollary 3.7.2 holds. To get around this problem we start with the construction of the minimal representation and prove Theorem 4.4.1 which states that the representations obtained via dual pair correspondence are quaternionic representations. More importantly the theorem gives an upper estimate for the quaternionic representations that could occur in the restriction. The crucial observation is that the correspondence for the dual pair $F_{4,4} \times \mathrm{SU}_2$ attains this upper bound! Then by using the method of see-saw pairs and computing K-types we get Theorem 12.4.1.

1.9 Theorem 4.4.1 could be sharpened to give better estimates. We will treat this in a different paper. In this paper we only do enough so as to prove the three correspondences.

1.10 The restrictions of the minimal representations of the exceptional groups to dual pairs of the form $G_{2,2} \times H$ where H is compact are studied in Huang–

Pandzic–Savin [HPS]. Also see [Li1] and [Li2] for other exceptional dual pairs correspondences.

1.11 Theorems 1.4.1, 12.2.1 and 12.4.1 lead to correspondences of the infinitesimal characters for the dual pairs. (See [Ho] §4(f), [Pr], [Li].) For the pairs $F_4 \times A_1$ and $D_4 \times A_1^3$, the correspondences are in agreement with Li [Li] and, Gross and Savin [GS] respectively. In [GS] Gross and Savin obtain the Θ-correspondence of the dual pair $\mathrm{SL}_2^3 \times \mathrm{Spin}(8)$ from the minimal representation of $E_{7,3}$. The correspondence in Theorem 1.4.1 is new.

ACKNOWLEDGEMENT: This paper is part of the author's Ph.D thesis at Harvard. He would like to thank his thesis advisor Prof. Benedict Gross for suggesting the problem. He is grateful to Prof. W. Schmid for pointing out Theorem 3.7.1. The discussion with G. Savin was very helpful. The author's research is generously supported by a scholarship from the National University of Singapore.

2. Exceptional groups of real rank 4

2.1 We refer to [GW2] §3 for the definition of a quaternionic real form G_0 of a complex linear group $G(\mathbb{C})$. It has maximal compact subgroup of the form

$$K = \mathrm{SU}_2 \times_{\mu_2} M \tag{6}$$

and we choose a positive root system Φ^+ such that the SU_2 above corresponds to the highest root $\tilde{\alpha}$. The Lie algebra $\mathfrak{g}_0$ of G_0 has Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}.$$

We define a self dual complex representation V_M of M by

$$\mathrm{Res}^K_{\mathrm{SU}_2 \times M}\ \mathfrak{p}(\mathbb{C}) = S^1_{\tilde{\alpha}} \otimes V_M. \tag{7}$$

V_M is a complex space of even dimension $2d$. $\mathfrak{p}$ is the tangent space of identity coset of G/K and $\mathrm{SU}_2(\tilde{\alpha}) \subset K$ acts on it as d copies of its action on the quaternion algebra $\mathbb{H}$. Thus this defines a quaternionic structure on G/K.

2.2 Let G_0 be one of the adjoint quaternionic groups $\mathrm{PSO}(4,4)$, $F_{4,4}$, $E_{6,4}$, $E_{7,4}$ and $E_{8,4}$ of real rank 4. We index them by $s = 0$, 1, 2, 4 and 8 respectively. We tabulate the highest weight λ of the $M(\mathbb{C})$ module V_M and its dimension $2d = 6s + 8$ below:

s	G_0	$M(\mathbb{C})$	V_M	λ	$2d$
0	$D_{4,4}$	$(\mathrm{SL}_2(A)\times\mathrm{SL}_2(B)\times\mathrm{SL}_2(C))/K_4$	$S_A^1\otimes S_B^1\otimes S_C^1$	–	8
1	$F_{4,4}$	$\mathrm{Sp}_6(\mathbb{C})$	$(\bigwedge^3\mathbb{C}^6)_0$	ϖ_3	14
2	$E_{6,4}$	$\mathrm{SL}_6(\mathbb{C})/\mu_3$	$\bigwedge^3\mathbb{C}^6$	ϖ_3	20
4	$E_{7,4}$	$\mathrm{Spin}_{12}(\mathbb{C})/\mu_2$	$\frac{1}{2}-\widetilde{\mathrm{Spin}}(4,4)$	ϖ_6	32
8	$E_{8,4}$	simply connected $E_7(\mathbb{C})$	minuscule	ϖ_7	56

2.3 There is a S_3 action on the $\mathfrak{g}_0$ which induces an automorphism of G_0 [L1]. The 4 exceptional quaternionic groups above contains the degree 4 cover $\widetilde{\mathrm{Spin}}(4,4)$ of $\mathrm{PSO}(4,4)$. The S_3 action stabilizes both $\widetilde{\mathrm{Spin}}(4,4)$ and its maximal compact subgroup as given in (4). It gives the outer automorphism group of $\widetilde{\mathrm{Spin}}(4,4)$, acts trivially on $\mathrm{SU}_2(\tilde{\alpha})$ but permutes the other three SU_2 in (4). For $E_{8,4}$ and $E_{7,4}$, the S_3 action also stabilizes the subgroups in (1), (2) and (3).

2.4 $H_1\times_C H_2$ in G_0 is called a **dual pair** if the centralizer of H_i is H_{i+1} for $i\in\mathbb{Z}/2\mathbb{Z}$. If $H_1'\times_{C'} H_2'$ is another dual pair in G_0 such that $H_1'\subset H_1$ and $H_2\subset H_2'$ then we call them a **see-saw pair** and write

$$\begin{array}{ccc} H_1 & & H_2' \\ & \bowtie & \\ H_1' & & H_2 \end{array}$$

2.5 The subgroups in (1), (2) and (3) are dual pairs in $E_{8,4}$ and $E_{7,4}$ respectively. Those in $E_{7,4}$ form the see-saw pair

$$\begin{array}{ccc} F_{4,4} & & \mathrm{SU}_2(A')\times\mathrm{SU}_2(B')\times\mathrm{SU}_2(C') \\ & \bowtie & \\ \widetilde{\mathrm{Spin}}(4,4) & & \mathrm{SU}_2 \end{array} \tag{8}$$

where SU_2 embeds diagonally into SU_2^3.

3. Quaternionic representations

3.1 Let G_0 be a quaternionic real group defined in §2.1 and let G be its double cover with maximal compact subgroup

$$K:=\mathrm{SU}_2(\tilde{\alpha})\times M.$$

In [GW1], Gross and Wallach constructed a series of representations π_k' of G from G-equivariant line bundles. A more general construction using G-equivariant vector bundles was studied by H. W. Wong [W1] [W2], who generalized the results of

Schmid [S1]. We will call these representations the quaternionic representations. In this section we will derive some properties of these representations.

3.2 Let $L := U_1 \times M \subset K$ and define the homogeneous space $D \equiv D(G) := G/L$. Define

$$\mathfrak{u} = \sum_{\langle\alpha,\tilde{\alpha}\rangle>0} \mathfrak{g}_\alpha, \qquad \overline{\mathfrak{u}} = \sum_{\langle\alpha,\tilde{\alpha}\rangle<0} \mathfrak{g}_\alpha,$$
$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}, \qquad \overline{\mathfrak{q}} = \mathfrak{l} + \overline{\mathfrak{u}}.$$

$\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are opposite parabolic subalgebras with Levi factor $\mathfrak{l} = \mathrm{Lie}(L)$. Since $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{l}$, this implies that D has a complex structure and $K/L = \mathrm{SU}_2/U_1 = \mathbb{P}^1(\mathbb{C})$ embeds as a projective curve into D.

Let $W[k] = e^{-k\frac{\tilde{\alpha}}{2}} \otimes W$ be a representation of $L = U_1 \times M$. We will denote the G-equivariant vector bundle constructed from $W[k]$ by $\mathcal{L}(W[k])$ or simply $\mathcal{L}$.

3.3 Suppose D' is a complex submanifold of D, we denote $\mathcal{F}^n_{D'}(\mathcal{L})$ as the holomorphic sheaf of germs vanishing of order at least n along D' and $\mathcal{N}_{D/D'}$ be the normal bundle of $D' \subset D$. Note that $\mathcal{F}^n_{D'}(\mathcal{L})$ is a coherent analytic sheaf and $\mathcal{N}_{D/D'}$ is supported on D'. Moreover there is an exact sequence

$$0 \to \mathcal{F}^{n+1}_{D'}(\mathcal{L}) \to \mathcal{F}^n_{D'}(\mathcal{L}) \to \mathcal{L} \otimes \mathrm{Sym}^n(\mathcal{N}^\vee_{D/D'}) \to 0. \tag{9}$$

We will denote the sheaf of holomorphic sections of $\mathcal{L}$ by $\mathcal{O}(\mathcal{L})$ and its Dolbeault complex over D by $(\Omega^{0,\bullet}(\mathcal{L}), \overline{\partial})$. Finally we will denote the sheaf cohomology $H^\bullet(D, \mathcal{O}(\mathcal{L}(W[k])))$ by $H^\bullet(D(G), W[k])$ or $H^\bullet(D, W[k])$. The left translation l_g action of G on $\Omega^{0,\bullet}(\mathcal{L})$ induces a G-action on the cohomology $H^\bullet(D, \mathcal{O}(\mathcal{L}))$. The next theorem summarizes the work of Schmid [S1], Wong [W1] [W2], and Gross and Wallach [GW1] [GW2].

THEOREM 3.3.1: *Let W be a finite dimensional irreducible representation of M with highest weight μ and $k \geq 2$ be an integer. Let $\mathcal{L} = \mathcal{L}(W[k])$ be the G-equivariant bundle. Then*

(a) *$(\Omega^{0,\bullet}(\mathcal{L}), \overline{\partial})$ has closed range property, that is, the image of $\overline{\partial}$ is closed. $H^\bullet(D, W[k])$ is a complex Frechet space;*
(b) *$H^i(D, W[k]) = 0$ for $i \neq 1$;*
(c) *$H^1(D, W[k])$ is an admissible representation of G of finite length. It has infinitesimal character $\mu + \rho(G) - k\frac{\tilde{\alpha}}{2}$ and it is the globalization of the Zuckerman module (see [V])*

$$\Gamma^1_{K/L}(\mathrm{Hom}_{\mathcal{U}(\overline{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), W[k])_L)$$

where $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ (see [V] for the definition of the Zuckerman functor);

(d) *$H^1(D, W[k])$ has K-types ($K = \mathrm{SU}_2(\tilde{\alpha}) \times M$)*

$$\bigoplus_{n=0}^{\infty} S_{\tilde{\alpha}}^{k-2+n} \otimes (\mathrm{Sym}^n(V_M) \otimes W);$$

(e) *$\mathcal{H}_n := H^1(D, \mathcal{F}^n_{\mathbb{P}^1}(\mathcal{L}))$ form an K-invariant filtration of $H^1(D, W[k])$. Moreover $X \cdot \mathcal{H}_{n+1} \subset \mathcal{H}_n$ for all $X \in \mathfrak{g}$ and they satisfy*

$$\text{(i)} \quad \mathcal{H}_n/\mathcal{H}_{n+1} \simeq H^1(\mathbb{P}^1, \mathcal{L} \otimes \mathrm{Sym}^n(\mathcal{N}^\vee_{D/\mathbb{P}^1})) \simeq S_{\tilde{\alpha}}^{k-2+n} \otimes \mathrm{Sym}^n(V_M) \otimes W,$$

$$\text{(ii)} \quad \textstyle\bigcap_n \mathcal{H}_n = 0;$$

(f) *$H^1(D, W[k])$ has Gelfand–Kirillov dimension* dim V_M + *1 and Bernstein degree* dim $W \cdot$ dim V_M.

(g) *There is a unique irreducible G-submodule of $H^1(D, W[k])$ which is generated by the translate of the 'lowest' K-types*

$$S_{\tilde{\alpha}}^{k-2} \otimes W.$$

Proof: (a) and (c). See Thm. 6 [W2].
(b) A result of Schmid and Wolf [SW] states that

$$H^i(D, \mathcal{S}) = 0 \tag{10}$$

for any coherent analytic sheaf $\mathcal{S}$ if $i \geq 2$. It remains to prove the case $i = 0$ and the proof is similar to that of Prop 5.7 [GW2].
(d) This follows by applying similar computations as in Sect. 9 [GW2] to the Zuckerman module in (c). When k is large, Prop. 54 [GW2] gives a direct verification.
(e) Part (i) follows by applying the long exact sequence to (9) with $D' = \mathbb{P}^1$. Part (ii) follows from (i) and (d).
(f) See Prop. 5.7 [GW2].
(g) Note that $H^0(\mathbb{P}^1, W[k] \otimes \wedge^q V_M[-1]) = 0$ for $q = 0, 1$ since k is assumed to be at least 2. Now proceed as in the proofs of Cor. 24, Cor. 29 [W1]. Also see Cor. 6.5 [S1]. ∎

3.4 By Proposition 3.2.12 of [V] and Theorem 3.3.1(d), every K-type of $H^1(D, W[k])$ has highest weight of the form $\omega + \sum_i \alpha_i$ where α_i is a non-compact positive root of G. This justifies the use of the word 'lowest' K-type.

3.5 To save on notations, we will denote the Harish-Chandra module of $H^1(D, W[k])$ by

$$\mathbf{H}(G, W[k]) = \mathbf{H}(W[k]) := \Gamma^1_{K/L}(\mathrm{Hom}_{\mathcal{U}(\bar{\mathfrak{q}})}(\mathcal{U}(\mathfrak{g}), W[k])_L).$$

Similarly $\sigma(G, W[k])$ or $\sigma(W[k])$ will denote the Harish-Chandra module of the unique irreducible submodule of $H^1(D, W[k])$. If the representation $H^1(D, W[k])$ descends to a representation of the quotient G_0 of G, then we may write it as $\mathbf{H}(G_0, W[k])$. The same applies to $\sigma(G_0, W[k])$. We will abuse notation and call $\mathbf{H}$ and σ representations of G. Furthermore if the representations are unitarizable, we will use the same symbols to denote the corresponding unitary representations of G.

We will call $H^1(D, W[k])$, its unique irreducible submodule or their Harish-Chandra modules **quaternionic** representation of G.

By Theorem 3.3.1(d) both $\mathbf{H}(G, W[k])$ and $\sigma(G, W[k])$ are $\mathrm{SU}_2(\tilde{\alpha})$-admissible.

If $\mathbf{H}(G, W[k])$ is unitarizable as a $(\mathfrak{g}, K)$-module, then it is irreducible and equals $\sigma(G, W[k])$. Indeed otherwise, the orthogonal complement of $\sigma(G, W[k])$ is a submodule which does not contain the lowest K-types and this contradicts Theorem 3.3.1(g).

3.6 We will determine which quaternionic representation is in the discrete series. To do this we will apply a result of Schmid which says that it suffices to check its K-types (see Corollary 3.7.2).

Recall that the K-types of a discrete series representation satisfy Blattner formula [HS]. First we will show that K-types of a quaternionic representation satisfy similar formula. Note that this is proven in Lemma 5.3 [S1] and Cor. 54 [W1] under certain 'negative' assumptions.

Let Φ^+, Φ^+_c and Φ^+_n be the sets of positive, positive compact and positive non-compact roots of G respectively, and let $\rho(G)$, ρ_c and ρ_n be the corresponding half sums of the roots in the sets. Let $W(K)$ be the Weyl group of the maximal compact subgroup K of G. Let

$$\Delta_G = \prod_{\alpha \in \Phi^+_n} (e^{-\frac{\alpha}{2}} - e^{\frac{\alpha}{2}}).$$

Define the formal character of K

$$\kappa \equiv \kappa(\mathbf{H}(D, W[k]) = \sum m_i \chi_i$$

of all the K-types of the quaternionic representation $\mathbf{H}(D, W[k])$.

LEMMA 3.6.1: *Let μ be the highest weight of the representation W of M and $\chi_M(W)$ be its character. Suppose $k \geq d+2$ where $d = \frac{1}{2}$ dim V_M, and let*

$$\Lambda_0 = (k-1-d)\frac{\tilde{\alpha}}{2} + \mu + \rho(M). \tag{11}$$

Then

$$\Delta_G \cdot \kappa = \frac{\sum_{\sigma \in W(K)} \operatorname{sgn}(\sigma)\, e^{\sigma(\Lambda_0)}}{\prod_{\alpha \in \Phi_c^+}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} = \chi_{\tilde{\alpha}}(k-2-d)\chi_M(W) \tag{12}$$

where $\chi_{\tilde{\alpha}}(n)$ is the character of $S^n_{\tilde{\alpha}}$.

Proof: Observe that we may say that W is the trivial representation. By (7) the set of non-compact roots Φ_n^+ is

$$\{\frac{1}{2}\tilde{\alpha} + w\colon w \text{ is a weight of } V_M\}.$$

Let $\eta = \tilde{\alpha}/2$;

$$\begin{aligned}
\Delta_G^{-1} &= e^{d\eta}\prod_w \left(1 - e^{\eta + w}\right)^{-1} = \sum_{n=0}^{\infty} e^{(d+n)\eta}\tau_n, \\
\Delta_G^{-1} &= e^{-d\eta}\prod_w \left(1 - e^{-\eta + w}\right)^{-1} = \sum_{n=0}^{\infty} e^{-(d+n)\eta}\tau_n,
\end{aligned}$$

where τ_n is the character of $\operatorname{Sym}^n(V_M)$. Hence

$$\begin{aligned}
\kappa &= \chi_{\tilde{\alpha}}(k-2-d)\Delta_G^{-1} \\
&= \frac{e^{(k-1-d)\eta}\Delta_G^{-1} - e^{-(k-1-d)\eta}\Delta_G^{-1}}{e^{\eta} - e^{-\eta}} \\
&= \sum_{n=0}^{\infty} \frac{e^{(k-1+n)\eta}\tau_n - e^{-(k-1+n)\eta}\tau_n}{e^{\eta} - e^{-\eta}} \\
&= \sum_{n=0}^{\infty} \chi_{\tilde{\alpha}}(k-2+n)\tau_n. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 3.6.2: *Assume the hypothesis of Lemma 3.6.1. In the case when Λ_0 is a dominant weight with respect to Φ^+ and regular with respect to Φ_c^+, then the K-types of $H^1(D, W[k])$ satisfy Blattner's formula.*

Proof: This follows from Hecht–Schmid [HS] Lemma 5.7 which states that (12) is equivalent to Blattner's formula. $\blacksquare$

3.7 In [S2], Schmid constructed a $\mathcal{U}(\mathfrak{g})$-module with the following universal property:

Let G be a real Lie group with maximal compact subgroup K such that K contains a maximal torus of G. Let π be an admissible $\mathcal{U}(\mathfrak{g})$-module such that π contains no irreducible K-types of highest weight $\Lambda - \rho_c + \rho_n - A_c - A_n$ other than $\Lambda - \rho_c + \rho_n$ where A_c stands for a sum of positive compact roots and A_n stands for a sum of distinct positive non-compact roots. Then

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(Q, \pi) \simeq \mathrm{Hom}_{\mathfrak{k}}(\pi_{\mathfrak{k}}(\Lambda - \rho_c + \rho_n), \pi).$$

This follows from Lemma 2.15, equation (3.3) and Lemmas 3.8 and 4.2 of [S2].

Furthermore, by Lemma 2.23 and Lemma 3.8 of [S2], the K-multiplicities in Q are bounded by those given by Blattner's formula. Let $\pi = \pi_\Lambda$ be the discrete series representation with infinitesimal character Λ and lowest K-type of highest weight $\Lambda - \rho_c + \rho_n$. By the universal property of Q, we conclude that $Q = \pi_\Lambda$. Hence

THEOREM 3.7.1 (Schmid, Hecht–Schmid):

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\pi_\Lambda, \pi) \simeq \mathrm{Hom}_{\mathfrak{k}}(\pi_{\mathfrak{k}}(\Lambda - \rho_c + \rho_n), \pi). \qquad \blacksquare$$

COROLLARY 3.7.2: *If π is a unitary representation with the same K-types as n copies of a discrete series representations π_Λ, then $\pi = n\pi_\Lambda$.*

Proof: Applying Theorem 3.7.1 and taking orthogonal complement

$$\pi = \pi_\Lambda \oplus \pi_\Lambda^\perp.$$

Since $\pi_\Lambda^\perp$ has the same K-types as $(n-1)\pi_\Lambda$, the corollary follows by induction on n. $\blacksquare$

4. A filtration of $H^1(D, W[k])$

4.1 We will show that there is a filtration of $H^1(D, W[k])$ very similar to the filtration which gives the K-types of $H^1(D, W[k])$ in [S1]. A corollary of this says that all unitary representations which we obtain via dual pair correspondence are unitary quaternionic representations. The author learned later that similar ideas have appeared in the restrictions of the holomorphic discrete series [Ma] [JV].

4.2 Suppose $G' \subset G$ are two quaternionic real groups containing the $\mathrm{SU}_2(\tilde{\alpha})$. We have $K' \subset K$, $M' \subset M$, $L \subset L'$, $\mathbb{P}^1 \subset D' \subset D$, V_M and $V_{M'}$.

Consider the short exact sequence of sheaves in (9) with $\mathcal{L} = \mathcal{L}(W[k])$. Let $V_{M/M'} = V_M/V_{M'}$ as a representation of M' so that the vector bundle $\mathcal{L} \otimes \mathrm{Sym}^n(\mathcal{N}_{D/D'})^\vee$ over D' is the G'-equivariant bundle induced from $(W \otimes \mathrm{Sym}^n(V_{M/M'}))[k+n]$. Define

$$\mathcal{H}'_n := H^1(D, \mathcal{F}^n_{D'}(\mathcal{L}(W[k])))$$

and consider the long exact sequence of (9). By Theorem 3.3.1(b) and (10), we get

$$0 \to \mathcal{H}'_{n+1} \to \mathcal{H}'_n \to H^1(D', (W \otimes \mathrm{Sym}^n(V_{M/M'}))[k+n]) \to 0.$$

Now we will state the main proposition which is the analogue of Theorem 3.3.1(e):

Proposition 4.2.1: *$\mathcal{H}'_n$ forms a filtration of $H^1(D, W[k])$ such that*

$$\text{(a)} \quad \mathcal{H}'_n/\mathcal{H}'_{n+1} \simeq H^1(D', (W \otimes \mathrm{Sym}^n(V_{M/M'}))[k+n]),$$
$$\text{(b)} \quad \textstyle\bigcap_n \mathcal{H}'_n = 0.$$

Proof: It remains to prove part (b) and we will show that $\bigcap \mathcal{H}'_n$ is a representation of G' without K'-finite vectors. Since $H^1(D, W[k])$ is $\mathrm{SU}_2(\tilde{\alpha})$-admissible, it is sufficient to show that the K'-types of $H^1(D, W[k])$ is the same as

$$\sum_{n=0}^{\infty} H^1(D', (W \otimes \mathrm{Sym}^n(V_{M/M'}))[k+n]). \tag{13}$$

Indeed the K'-types of (13) must contain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} S_{\tilde{\alpha}}^{k+n-2+m} \otimes \mathrm{Sym}^m(V_{M'}) \otimes \mathrm{Sym}^n(V_{M/M'}) \otimes W \\
&\quad = \sum_{p=0}^{\infty} S_{\tilde{\alpha}}^{k-2+p} \otimes W \otimes \left(\sum_{p=m+n} \mathrm{Sym}^m(V_{M'}) \otimes \mathrm{Sym}^n(V_{M/M'})\right) \\
&\quad = \sum_{p=0}^{\infty} S_{\tilde{\alpha}}^{k-2+p} \otimes W \otimes \mathrm{Sym}^p(V_M)
\end{aligned}$$

which by Theorem 3.3.1(e) equals the K'-types of $H^1(D, W[k])$. ∎

COROLLARY 4.2.2: *If $\mathbf{H}(G, W[k])$ is unitarizable, then so are*

$$\mathbf{H}(G', (W \otimes \mathrm{Sym}^n(V_{M/M'}))[k+n])$$

and

$$\mathbf{H}(G, W[k]) = \bigoplus_{n=0}^{\infty} \mathbf{H}(G', (\mathrm{Sym}^n(V_{M/M'}) \otimes W)[k+n])$$

as a representation of G'. ∎

4.3 Let G be the double cover of one of the 4 quaternionic exceptional groups. We will describe the minimal representation π_Z of G.

In [GW1], Gross and Wallach showed that $\mathbf{H}(G, \mathbb{C}[k])$ is unitarizable for $k \geq d$. There are 3 additional values of $k < d$ where $\sigma(G, \mathbb{C}[k])$ is unitarizable, namely, $k = s+2$, $2s+2$, $3s+3$, and the 3 unitary representations are denoted by π_Z, π_Y and π_X, respectively. Here $Z \subset Y \subset X$ are the closure of the 3 orbits of $M(\mathbb{C})$ in $\mathbb{P}V_M(\mathbb{C})$. The K-types ($K = \mathrm{SU}_2(\alpha) \times M$) of π_Z are

$$\pi_Z = \bigoplus_{n=0}^{\infty} S_{\tilde{\alpha}}^{s+n} \otimes \pi_M(n\lambda) \tag{14}$$

where λ is the highest weight of V_M. The K-types of π_X and π_Y are given in Prop. 2.2 [GW1].

The annihilator of π_Z is precisely the Joseph ideal in the universal enveloping algebra of Lie(G) so it is also called the **minimal** representation of G.

4.4 Let $G' = H_1 \times_C H_2$ be a dual pair where H_2 is compact and let S be an irreducible representation of H_2. We define the unitary representation $\Theta(S)$ of H_1 by

$$\mathrm{Res}_{G'}^{G} \pi_Z = \bigoplus_{S} \Theta(S) \otimes S.$$

THEOREM 4.4.1: *Given a dual pair $H_1 \times_C H_2$ in G where H_1 is a quaternionic subgroup, that is, it contains $\mathrm{SU}_2(\tilde{\alpha})$ and $H_2 \subset M$. Then $\Theta(S)$ is a sum of quaternionic unitary representations of H_1.*

Suppose $\mathrm{Sym}^n(V_M) = \sum_i W_{i,n} \otimes W'_{i,n}$ as irreducible representations of $(H_1 \cap M) \times H_2$; then

$$\sum_S \Theta(S) \otimes S \subset \sum_{n=0}^{\infty} \sum_i \sigma(H_1, W_{i,n}[n+s]) \otimes W'_{i,n}.$$

Proof: Let H'_n be the Harish-Chandra module of $\mathcal{H}'_n$ in Proposition 4.2.1. Since π_Z is $\mathrm{SU}_2(\alpha)$ admissible, π_Z decomposes discretely as a $H_1 \times_C H_2$ module. Hence any irreducible component of $\Theta(S) \otimes S$ is a submodule of $\oplus(H'_n/H'_{n+1})$. The theorem follows from Proposition 4.2.1(a) and Theorem 3.3.1(g). ∎

Since these representations are $\mathrm{SU}_2(\tilde{\alpha})$-admissible, the K-types of $\Theta(S)$ are of the form

$$\bigoplus_{n \geq n_0} S^n_{\tilde{\alpha}} \otimes W_n.$$

By Theorem 3.3.1(g) and §3.4, we are justified in calling $S^{n_0}_{\tilde{\alpha}} \otimes W_{n_0}$ the lowest K-type of $\Theta(S)$.

4.5 For the dual pair $\mathrm{Spin}(4,4) \times_{K_4} G_s$, $\Theta(S)$ is a direct sum of representations of the form

$$\sigma = \sigma(\mathrm{Spin}(4,4), S^a_A \otimes S^b_B \otimes S^c_C[k]).$$

The S_3 action (cf. §2.7) operates on the minimal representation and thus induces a symmetry on the dual pair correspondence which we will explore in the later proofs. Let $s \in S_3$; then

$$s(\sigma) = \sigma(\mathrm{Spin}(4,4), S^a_{s(A)} \otimes S^b_{s(B)} \otimes S^c_{s(C)}[k]).$$

4.6 Finally we produce some unitary representations needed in later proofs. $\mathbf{H}(\tilde{F}_{4,4}, \mathbb{C}[k])$ is unitarizable if $k \geq 7$ (cf. [GW2]) so by Corollary 4.2.2

$$\begin{aligned} &\mathrm{Res}^{\tilde{F}_{4,4}}_{\widetilde{\mathrm{Spin}}(4,4)} \mathbf{H}(\tilde{F}_{4,4}, \mathbb{C}[k]) \\ &= \bigoplus_{n=0}^{\infty} \mathbf{H}(\widetilde{\mathrm{Spin}}(4,4), (\mathrm{Sym}^n(\mathbb{C}^2_A + \mathbb{C}^2_B + \mathbb{C}^2_C))[k+n]) \\ &= \bigoplus_{n=0}^{\infty} \bigoplus_{a+b+c=n} \mathbf{H}(\widetilde{\mathrm{Spin}}(4,4), (S^a_A \otimes S^b_B \otimes S^c_C)[k+n]). \end{aligned}$$

Therefore

$$\mathbf{H}(\widetilde{\mathrm{Spin}}(4,4), (S^a_A \otimes S^b_B \otimes S^c_C)[a+b+c+k]) \tag{15}$$

is unitarizable. If $k \geq 9$ it is the discrete series representation by Corollary 3.7.2.

PROPOSITION 4.6.1: *Let $S(a,b,c) = S^a_A \otimes S^b_B \otimes S^c_C$. Then*

$$\mathrm{Res}^{F_{4,4}}_{\mathrm{Spin}(4,4)} \pi_X = \bigoplus_{a,b,c} \sigma(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]).$$

When a,b,c are strictly positive, the following sequence is exact:

$$\begin{aligned} 0 &\to \sigma(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]) \to \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]) \to \\ &\to \mathbf{H}(\mathrm{Spin}(4,4), S(a-1,b-1,c-1)[7+a+b+c]) \to 0. \end{aligned} \tag{16}$$

Otherwise, if at least one of a, b or c is zero, then

(17) $\sigma(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]) = \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c])$.

Proof: Proposition 8.5 of [GW2] states that the sequence

$$0 \to \pi_X \to \mathbf{H}(F_{4,4}, \mathbb{C}[6]) \to \mathbf{H}(F_{4,4}, \mathbb{C}[10]) \to 0$$

is exact and $\mathbf{H}(F_{4,4}, \mathbb{C}[10])$ is unitarizable.

We claim that the sequence

$$\begin{aligned} 0 \to \pi_X \to & \bigoplus_{a,b,c} \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]) \to \\ \to & \bigoplus_{a,b,c} \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[10+a+b+c]) \to 0 \end{aligned} \tag{18}$$

is exact as representations of Spin(4,4). Indeed, we apply the filtration $\mathcal{H}'_n$ in Proposition 4.2.1 to get

$$\mathcal{H}'_n \to \bigoplus_{a,b,c} \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[10+a+b+c]).$$

By (15) each summand in the image is irreducible, and we get

$$\mathcal{H}'_n/\mathcal{H}'_{n+1} \to \bigoplus_{\text{some } a,b,c} \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[10+a+b+c]) \to 0.$$

The claim follows from Proposition 4.2.1.

Now we will break up the exact sequence (18) using infinitesimal characters of Spin(4,4). This gives (16) and (17) for different infinitesimal characters. ∎

5. Dual pair correspondence of $E_{8,4}$

5.1 Recall that the minimal representation π_Z of $E_{8,4}$ has K-types ($K = \mathrm{SU}_2 \times_{\mu_2} E_7$)

$$\bigoplus_{N=0}^{\infty} S^{N+8}(\mathbb{C}^2) \otimes \pi(N\lambda),$$

where λ is the highest weight of the miniscule representation of E_7 of 56 dimension.

5.2 We will identify the various subalgebras of $E_{8,4}$.

Let $\tilde{\alpha}$ be the highest root of $E_{8,4}$ and Spin(4,4). We will denote the set of positive roots for the compact groups $E_6 \subset E_7$ as:

$$E_7: \quad \varepsilon_8 \pm \varepsilon_7, \quad \varepsilon_i \pm \varepsilon_j, \quad 1 \le i < j \le 6,$$

$$\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^{6} \pm\varepsilon_i); \tag{19}$$

$$E_6: \quad \varepsilon_i \pm \varepsilon_j \quad 2 \le i < j \le 6,$$

$$\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_1 - \sum_{i=2}^{6} \pm\varepsilon_i) \tag{20}$$

where there are an even number of positive signs in the summations in (19) and (20). We define the Lie subgroups $D_n \subset E_7$ ($n = 4, 6$) as those having positive roots

$$\varepsilon_i \pm \varepsilon_j, \quad 1 \le i < j \le n.$$

Note that the notations differ slightly from those given in Bourbaki [Bou1] Planche IV, V, VI for the root systems of E_7 and E_6.

5.3 Define

$$\eta = \frac{\tilde{\alpha}}{2}, \qquad A = \exp\Big(\frac{\varepsilon_8 - \varepsilon_7}{2}\Big),$$
$$B = \exp\Big(\frac{\varepsilon_5 - \varepsilon_6}{2}\Big), \qquad C = \exp\Big(\frac{\varepsilon_5 + \varepsilon_6}{2}\Big).$$

The Lie subgroup SU_2 corresponding to $\tilde{\alpha}$, A, B, C lies in the maximal compact subgroup K of Spin(4,4) and coincides with those given in (4).

6. Branching rules I

6.1 In this section, the Lie subgroup $D_5 \subset D_6$ is the Lie subgroup defined by the positive roots $\varepsilon_i \pm \varepsilon_j$ for $2 \le i < j \le 6$ (cf. §5.2).

In [G1], B. Gross states the following branching rule:

LEMMA 6.1.1:

$$\mathrm{Res}^{E_7}_{\mathrm{SU}_2 \times \mathrm{Spin}(12)} \pi(N\lambda) = \bigoplus_{a+2b+c=N} S^c_A(\mathbb{C}^2) \otimes \pi(a\varpi_6 + b\varpi_2 + c\varpi_1).$$

We will need this lemma in §7. Since we could not find a proof of it in the literature, we will devote this section to proving it. We prefer to work on the level of Lie algebras.

6.2 First we prove a lemma that is very useful in deriving branching rules based on an idea from [HPS]:

LEMMA 6.2.1: *Suppose $\pi(\Omega_1)$ and $\pi(\Omega_2)$ occur in the restriction of $\pi(N_1\lambda)$ and $\pi(N_2\lambda)$, respectively. Then $\pi(\Omega_1+\Omega_2)$ occurs in $\pi((N_1+N_2)\lambda)$. Moreover, if the multiplicity of $\pi(\Omega_1+\Omega_2)$ is 1, then so are $\pi(\Omega_1)$ and $\pi(\Omega_2)$.*

Proof: By the Borel–Weil Theorem, there exists a holomorphic line bundle $\mathfrak{L}$ over the complex manifold E_7/T such that the global section $\Gamma(E_7/T, \mathfrak{L}^{\otimes N})$ is the representation $\pi(N\lambda)$. Suppose $v_i \in \Gamma(E_7/T, \mathfrak{L}^{\otimes N_i})$ $(i=1,2)$ is the highest weight for $\pi(\Omega_i)$; then

$$v_1 \otimes v_2 \in \Gamma(E_7/T, \mathfrak{L}^{\otimes(N_1+N_2)})$$

is a highest weight for $\pi(\Omega_1+\Omega_2)$; $v_1\otimes v_2$ is nonzero as v_1 and v_2 are holomorphic. ∎

6.3 Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{e}_7$. We align the Cartan subalgebras and choose a positive root system so that a positive root space of $\mathfrak{g}$ is also a positive root space of $\mathfrak{e}_7$. We will use $\rho(\mathfrak{g})$ to denote the half sum of the positive roots. Let $\rho_7 = \rho(\mathfrak{e}_7)$ and $\rho_6 = \rho(\mathfrak{e}_6)$ so that

$$\begin{aligned} \lambda &= \varepsilon_1 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7), \\ \rho_7 - \rho_6 &= 9\lambda. \end{aligned} \tag{21}$$

Let Φ^+ be the set of positive roots of $\mathfrak{g}$. Then

$$\sum_{\sigma\in W(\mathfrak{g})} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho(\mathfrak{g}))} = e^{\rho(\mathfrak{g})} \prod_{\alpha\in\Phi(\mathfrak{g})} (1-e^{-\alpha}). \tag{22}$$

This is a simple corollary of the Weyl Character Formula (WCF) which we will use repeatedly. (See Fulton–Harris [FH] Lemma 24.3.)

6.4 Assume that

$$\operatorname{Res}^{E_7}_{\mathrm{SU}_2\times \mathrm{Spin}(12)} \pi(N\lambda) = \bigoplus_{c,\Omega} m(c,\Omega)\, S^c_A(\mathbb{C}^2) \otimes \pi(\Omega). \tag{23}$$

where $\Omega = r_1\varpi_1 + \cdots + r_6\varpi_6$ and $m(c,\Omega)$ denotes the multiplicity of the representation.

When $N=1$ and 2, we can verify Lemma 6.1.1 from the table given in McKay–Patera [KP]. Lemma 6.2.1 shows that the left hand side of Lemma 6.1.1 contains its right hand side. We claim that it is sufficient to show that Lemma 6.1.1 holds for r_1, c large compared to $r_2, \dots, r_6$. Indeed, this follows from Lemma 6.2.1 by observing that $S^1_A(\mathbb{C}^2) \otimes \pi(\varpi_1)$ occurs once in the restriction of $\pi(\lambda)$. The above claim is not absolutely essential but it helps to simplify matters.

6.5 PROOF OF LEMMA 6.1.1: Taking characters on both sides of (23), applying the WCF and clearing the denominators on the right hand side of the equation, we get

$$\begin{aligned}&\sum_{\sigma\in W(E_7)} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho_7+N\lambda)} \frac{e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)} \prod_{\mathfrak{su}_2\oplus\mathfrak{d}_6}(1-e^{-\alpha})}{e^{\rho_7}\prod_{\mathfrak{e}_7}(1-e^{-\alpha})}\\ &= \sum_{c,\Omega} m(c,\Omega)(A^{c+1}-A^{-c-1}) \sum_{\sigma\in W(E_6)} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho(\mathfrak{d}_6)+\Omega)}.\end{aligned} \tag{24}$$

With reference to Cartan [Ca] eqn. (24), p. 273, the 56 weights of λ divide into 2 orbits under the action of $W(\mathrm{SU}_2 \times D_6)$ generated respectively by

$$\lambda = \varepsilon_1 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7) \quad \text{and} \quad \varpi_6 + \frac{1}{2}(\varepsilon_8-\varepsilon_7) = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_6) + \frac{1}{2}(\varepsilon_8 - \varepsilon_7).$$

Hence in the orbit of 56 weights of $\rho_7 + N\lambda$ under the Weyl group $W(E_7)$, only the weight $\rho_7 + N\lambda$ can contribute to large r_1 and c on the right hand side of (24). Let

$$\Sigma_0 = \sum_{\sigma\in W(E_6)} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho_7+N\lambda)} \frac{e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)} \prod_{\mathfrak{su}_2\oplus\mathfrak{d}_6}(1-e^{-\alpha})}{e^{\rho_7}\prod_{\mathfrak{e}_7}(1-e^{-\alpha})} \tag{25}$$

be the partial sum in the left hand side of (24). The above discussion shows that *the power series expansion of* Σ_0 *is*

$$\sum_{c,\Omega} m(c,\Omega)\, A^{c+1} \sum_{\sigma\in W(D_5)} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho(\mathfrak{d}_6)+\Omega)} \tag{26}$$

if r_1, c *is large compared to* $r_2, \dots, r_6$.

In general it is very tedious to compute the action of $W(E_6)$ on a weight. The key observation to get around this is (21) and the fact that the stabilizer of λ in the Weyl group of E_7 is the Weyl group of E_6. Therefore we can apply (22).

$$\begin{aligned}\Sigma_0 &= e^{\rho_7-\rho_6+N\lambda} \sum_{\sigma\in W(E_6)} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho_6)} \frac{e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)} \prod_{\mathfrak{su}_2\oplus\mathfrak{d}_6}(1-e^{-\alpha})}{e^{\rho_7}\prod_{\mathfrak{e}_7}(1-e^{-\alpha})}\\ &= e^{\rho_7-\rho_6+N\lambda}\left(e^{\rho_6}\prod_{\mathfrak{e}_6}(1-e^{-\alpha})\right) \frac{e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)} \prod_{\mathfrak{su}_2\oplus\mathfrak{d}_6}(1-e^{-\alpha})}{e^{\rho_7}\prod_{\mathfrak{e}_7}(1-e^{-\alpha})}\\ &= e^{N\lambda+\rho(\mathfrak{su}_2)} \frac{e^{\rho(\mathfrak{d}_6)}\prod_{\mathfrak{d}_5}(1-e^{-\alpha})}{\prod'(1-e^{-\alpha})}.\end{aligned} \tag{27}$$

In the above equality, the product $\prod'$ is taken over roots of the form

$$\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \varepsilon_1 + \sum_{i=2}^{6} \pm\varepsilon_i)$$

where there is an odd number of positive signs in the summation.

Let

$$R = \exp\left(\frac{1}{2}\sum_{i=1}^{6}\varepsilon_i\right) = e^{\varpi_6};$$

then

$$\begin{aligned}\left(\prod{}'(1-e^{-\alpha})\right)^{-1} &= \sum_{\sigma\in N_5}\sigma\left(\frac{1}{\prod''(1-e^{-\beta})}\frac{1}{1-e^{-\lambda}R}\right)\\ &= \sum_{\sigma\in N_5}\sigma\left(\frac{1}{\prod''(1-e^{-\beta})}\sum_{n=0}^{\infty}e^{-n\lambda}R^n\right),\end{aligned} \tag{28}$$

where N_5 is the normal subgroup $\langle\pm1\rangle_{\Pi=1}^{5}$ of the Weyl group $W(D_5)$ and the product $\prod''_\beta$ is taken over weights β of the form

$$\varepsilon_i + \varepsilon_j, \quad \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$$

where i, j, k, l are distinct integers between 2 and 6 inclusively.

Substituting (28) into (27), we get

$$\Sigma_0 = e^{N\lambda+\rho(\mathfrak{su}_2)}\sum_{\sigma\in N_5}\operatorname{sgn}(\sigma)\,\sigma\left(\frac{e^{\rho(\mathfrak{d}_6)}\prod'''(1-e^{-\alpha})}{\prod^{(iv)}(1-e^{-\beta})}\sum_{n=0}^{\infty}e^{-n\lambda}R^n\right), \tag{29}$$

where the product $\prod'''$ is taken over roots α of the form

$$\varepsilon_i - \varepsilon_j, \quad 2 \le i < j \le 6$$

and the product $\prod^{(iv)}$ is taken over weights β of the form

$$\varepsilon_{\overline{\sigma}(3)} + \cdots + \varepsilon_{\overline{\sigma}(6)} = 2\varpi_6 - \overline{\sigma}(\varpi_2)$$

where $\overline{\sigma}$ is a permutation of the set $\{2, \ldots, 6\}$.

Similar to what we did in (28), we set

$$\begin{aligned} T &= e^{\varepsilon_1+\varepsilon_2} = e^{\varpi_1},\\ U &= \{1, (23), (24), (25), (26)\} \subset W(D_5),\end{aligned}$$

and we get

$$\frac{e^{\rho(\mathfrak{d}_6)}\prod'''(1-e^{-\alpha})}{\prod^{(iv)}(1-e^{-\beta})} = \sum_{\overline{\sigma}\in U}\operatorname{sgn}(\overline{\sigma})\,\overline{\sigma}\left(e^{\rho(\mathfrak{d}_6)}\prod^{(v)}(1-e^{-\alpha})\sum_{m=0}^{\infty}(TR^{-2})^m\right)$$
$$= \sum_{\overline{\sigma}\in U}\operatorname{sgn}(\overline{\sigma})\,\overline{\sigma}\left(\sum_{\sigma\in S_4}\operatorname{sgn}(\sigma)\,e^{\sigma(\rho(\mathfrak{d}_6))}\sum_{m=0}^{\infty}(TR^{-2})^m\right)$$
$$(30)\qquad = \sum_{\sigma\in S_5}\operatorname{sgn}(\sigma)\,\sigma\left(e^{\rho(\mathfrak{d}_6)}\sum_{m=0}^{\infty}(TR^{-2})^m\right)$$

where the product $\prod^{(v)}$ is taken over roots α of the form

$$\varepsilon_i-\varepsilon_j,\quad 3\le i<j\le 6.$$

Finally, putting (30) into (29), we have

$$(31)\qquad \Sigma_0 = e^{N\lambda+\rho(\mathfrak{su}_2)}\sum_{\sigma\in W(D_5)}\operatorname{sgn}(\sigma)\,\sigma\left(e^{\rho(\mathfrak{d}_6)}\sum_{m=0}^{\infty}(TR^{-2})^m\sum_{n=0}^{\infty}e^{-n\lambda}R^n\right).$$

We are interested in those exponents e^{ε} where $\varepsilon=(\varepsilon_1>\cdots>|\varepsilon_6|)$ that lies in the Weyl chamber of $\mathfrak{d}_6$. The sum of those exponents in (31) is

$$e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)}\sum_{n=0}^{N}\sum_{m=0}^{[n/2]}e^{(N-n)\lambda}T^mR^{n-2m}$$
$$= e^{\rho(\mathfrak{su}_2\oplus\mathfrak{d}_6)}\sum_{a+2b+c=N}A^ce^{a\varpi_6+b\varpi_2+c\varpi_1}$$

and by comparing with (26) completes the proof of Lemma 6.1.1. ∎

7. Branching rules II

7.1 In this section we will derive the branching rule for the representation $\pi(N\lambda)$ of E_7 when restricted to $\mathrm{Spin}(8)\times_{K_4}\mathrm{SU}_2^3$. The main result of this section is

PROPOSITION 7.1.1:

$$\operatorname{Res}^{E_7}_{\mathrm{Spin}(8)\times_{K_4}\mathrm{SU}_2^3}\pi(N\lambda)$$
$$= \bigoplus_{\Omega}(n_2+1)\ \pi_{\mathrm{Spin}(8)}(\Omega)\otimes$$
$$\otimes\left(S_A^{n_1}\otimes S_B^{n_2}\otimes S_C^{n_3}\otimes\operatorname{Sym}^{N-(n_1+2n_2+n_3+n_4)}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)\right)$$

where the sum is taken over all dominant weights $\Omega = n_1\varpi_1 + \cdots + n_4\varpi_4$ *of* Spin(8) *such that* $N \geq n_1 + 2n_2 + n_3 + n_4$.

First we recall a well known branching rule (see Harris–Fulton [FH] p. 426 or Boerner [Bo]):

THEOREM 7.1.2: (a) *Let* $\mathfrak{d}_n \subset \mathfrak{b}_n$ *and* $\pi(\beta)$ *be the representation of* $\mathfrak{b}_n$ *with highest weight* $\beta = (\beta_1 \geq \cdots \geq \beta_n \geq 0)$. *Then*

$$\mathrm{Res}^{\mathfrak{d}_n}_{\mathfrak{b}_n}(\pi(\beta)) = \bigoplus \pi(\xi)$$

where the sum is taken over all $\xi = (\xi_1 \geq \cdots \geq |\xi_n|)$ *satisfying*

$$\beta_1 \geq \xi_1 \geq \beta_2 \geq \cdots \geq \xi_{n-1} \geq \beta_n \geq |\xi_n|,$$

with the β_i *and* ξ_i *simultaneously all integers or all half integers.*

(b) *Let* $\mathfrak{b}_{n-1} \subset \mathfrak{d}_n$ *and* $\pi(\beta)$ *be the representation of* $\mathfrak{d}_n$ *with highest weight* $\beta = (\beta_1 \geq \cdots \geq |\beta_n|)$. *Then*

$$\mathrm{Res}^{\mathfrak{d}_n}_{\mathfrak{b}_{n-1}}(\pi(\beta)) = \bigoplus \pi(\xi)$$

where the sum is taken over all $\xi = (\xi_1 \geq \cdots \geq \xi_{n-1} \geq 0)$ *satisfying*

$$\beta_1 \geq \xi_1 \geq \beta_2 \geq \cdots \geq \xi_{n-1} \geq |\beta_n|,$$

with the β_i *and* ξ_i *simultaneously all integers or all half integers.*

We will prove Proposition 7.1.1 by applying Lemma 6.1.1 and then further restrict the representation $\pi(a\varpi_6 + b\varpi_2 + c\varpi_1)$ of Spin(12) to the subgroup $\mathrm{SU}_2^2 \times \mathrm{Spin}(8)$. We prefer to work on the level of Lie algebras.

7.2 Recall §5.2 and let $(\varepsilon_1, \ldots, \varepsilon_6)$ correspond to the Cartan Subalgebra (CSA) of $\mathfrak{d}_6$. Let $(\varepsilon_1, \ldots, \varepsilon_4)$ correspond to the CSA of $\mathfrak{d}_4$, and $\varepsilon_5 - \varepsilon_6$ and $\varepsilon_5 + \varepsilon_6$ correspond to the CSA's of $\mathfrak{su}_2(B)$ and $\mathfrak{su}_2(C)$, respectively (cf. (21)).

Let

$$\rho = (5,4,3,2,1,0) \quad \text{and} \quad \overline{\rho} = (3,2,1,0,0,0),$$

respectively, be the half sums of the positive roots of $\mathfrak{d}_6$ and $\mathfrak{d}_4$.

Put $t_i = e^{\varepsilon_i}$. We define a power series $\Psi(x)$ and its coefficients $f_n(t_5, t_6)$ for $n \in \mathbb{Z}$ by the following formula:

$$\Psi(x) = \prod_{j=5,6} (1 - xt_j)^{-1}(1 - xt_j^{-1})^{-1} = \sum_{n=-\infty}^{\infty} f_n x^n. \tag{32}$$

Note that f_n are rational functions in t_5 and t_6, $f_0 = 1$ and $f_n = 0$ for $n < 0$ and f_n is the character of $\mathrm{Sym}^n(\mathbb{C}_B^2 \otimes \mathbb{C}_C^2)$ for $n \geq 0$ (cf. (21)). These functions will be useful in the later proofs.

7.3 We will use

$$\Lambda = a\varpi_6 + b\varpi_2 + c\varpi_1$$

to denote a weight of $\mathfrak{d}_6$ throughout this section.

Let

$$\Omega = (\xi_1 \geq \xi_2 \geq \xi_3 \geq |\xi_4|) = n_1\varpi_1 + n_2\varpi_2 + n_3\varpi_3 + n_4\varpi_4 \tag{33}$$

be a dominant weight of $\mathfrak{d}_4$. Here ξ_i are half integers and ϖ_i are the fundamental weights of $\mathfrak{d}_4$ and

$$\begin{aligned} n_1 &= \xi_1 - \xi_2, & n_2 &= \xi_2 - \xi_3, \\ n_3 &= \xi_3 - \xi_4, & n_4 &= \xi_3 + \xi_4. \end{aligned}$$

Suppose the representation $\pi_{\mathfrak{d}_4}(\Omega)$ appears in the restriction of $\pi_{\mathfrak{d}_6}(\Lambda)$ to the subalgebra $\mathfrak{d}_4$. By applying Theorem 7.1.2 four times, we see that it is both necessary and sufficient for Ω to satisfy

$$\frac{a}{2} + b + c \geq \xi_1 \geq \frac{a}{2}, \quad \frac{a}{2} + b \geq \xi_2 \geq \frac{a}{2}, \tag{34}$$

$$\frac{a}{2} \geq \xi_3 \geq |\xi_4|, \quad a \equiv 2\xi_i \pmod 2. \tag{35}$$

Now suppose the $\pi_{\mathfrak{d}_4}(\Omega)$ appears in the restriction of $\pi_{\mathfrak{e}_7}(N\lambda)$. We claim that it is both necessary and sufficient for Ω to satisfy

$$N \geq n_1 + 2n_2 + n_3 + n_4.$$

Indeed necessity is clear. To show sufficiency, we have to produce a, b, c that satisfy $c + 2b + a = N$ and (34) to (35). One can check that $a = n_3 + n_4$, $b = n_2$ and $c = N - 2n_2 - n_3 - n_4$ will do. This proves the claim.

7.4 Suppose

$$\mathrm{Res}^{D_6}_{D_4 \times \mathrm{SU}_2(B) \times \mathrm{SU}_2(C)} \pi(\Lambda) = \sum_i \pi(\Omega_i) \otimes \pi\Big(n_i \frac{\varepsilon_5 - \varepsilon_6}{2}\Big) \otimes \pi\Big(n_i' \frac{\varepsilon_5 + \varepsilon_6}{2}\Big).$$

Taking characters on both sides and applying the WCF and clearing denominators,

$$\frac{\sum_{\sigma \in W(D_6)} \mathrm{sgn}(\sigma)\, e^{\sigma(\rho + \Lambda)}}{\prod_{i,j}{}'(1 - t_i^{-1} t_j^{\pm 1})} = e^{\rho - \overline{\rho}} \sum_i \sum_{\overline{\sigma} \in W(D_4)} \mathrm{sgn}(\overline{\sigma})\, e^{\overline{\sigma}(\overline{\rho} + \Omega_i)} \cdot \chi_i(B, C) \tag{36}$$

where $t_i = e^{\epsilon}$ and the product $\prod'_{i,j}$ is taken over

$$(i,j) \in (\{1,\ldots,4\} \times \{5,6\}) \bigcup \{(5,6)\}$$

and $\chi_i(\cdot,\cdot)$ is the character of $\mathrm{SU}_2(B) \times \mathrm{SU}_2(C)$. We need to deduce χ_i.

7.5 The Weyl group of $\mathfrak{d}_4$ acts transitively on its Weyl chambers so χ_i is the coefficient of $e^{\Omega_i+\bar{\rho}}$. We will equate χ_i with the left hand side of the equation.

The left hand side of (36) can be written as

$$\sum_{\sigma\in W} \mathrm{sgn}(\sigma)\, e^{\sigma(\rho+\Lambda)} \prod_{i,j}{}' \left(\sum_{n=0}^{\infty} (t_i^{-1}t_j^{\pm})^n\right).$$

This shows that if $e^{\rho+\Lambda}$ where $\Lambda = (\beta_1 \geq \cdots \geq |\beta_6|)$ on the left hand side of (36) contributes to $e^{\rho+\xi}$ where $\xi = (\xi_1 \geq \cdots \geq |\xi_4|)$ on the right, then $\beta_i \geq \xi_i$ for i = 1, 2, 3, 4. With reference to (34) and (35), we see that such exponentials can only come from the sub sum

$$\sum_{\sigma\in W'} \mathrm{sgn}(\sigma)\, e^{\sigma(\rho+\Lambda)} \cdot \prod_{i,j}{}' \left(\sum_{n=0}^{\infty} (t_i^{-1}t_j^{\pm})^n\right) \tag{37}$$

where $W' \subset W(D_6) = \langle \pm 1\rangle_{\Pi=1}^{6} \rtimes S_6$ is the subgroup

$$W' = S_2 \times \{\langle 1,1,1,1,-1,-1\rangle \rtimes S_4\}.$$

We can decompose (37) into 2 factors, each coming from a factor subgroup of W':

$$\mathrm{Eqn}(37) = t_5 \cdot \mathbf{I} \cdot \mathbf{J} \tag{38}$$

where

$$\begin{aligned}
\mathbf{I} &= \left(t_1^{5+b+c+\frac{a}{2}} t_2^{4+b+\frac{a}{2}} - t_1^{4+b+\frac{a}{2}} t_2^{5+b+c+\frac{a}{2}}\right) \Psi(t_1^{-1})\Psi(t_2^{-1}),\\
\mathbf{J} &= \frac{1}{t_5} \sum_{\sigma\in S_4} \mathrm{sgn}(\sigma) \left\{\exp\left(\sigma\left(0,0,3+\frac{a}{2},2+\frac{a}{2},1+\frac{a}{2},\frac{a}{2}\right)\right)\right.\\
&\qquad \left. + \exp\left(\sigma\left(0,0,3+\frac{a}{2},2+\frac{a}{2},-1-\frac{a}{2},-\frac{a}{2}\right)\right)\right\}\\
&\qquad \times \Psi(t_3^{-1})\Psi(t_4^{-1}) \left(1-\frac{t_5}{t_6}\right)^{-1} \left(1-\frac{1}{t_5t_6}\right)^{-1}.
\end{aligned}$$

Let $\Omega_i = \Omega$ be a dominant weight of $\mathfrak{d}_4$ as in (33) and we will compute the corresponding χ_i. First we calculate the contributions from $\mathbf{I}$, that is, the coefficient of the term $t_1^{\xi_1+5}t_2^{\xi_2+4}$. Let

$$r = b + \frac{a}{2} - \xi_2, \tag{39}$$

$$n' = N - \xi_1 - \xi_2 = N - (n_1 + 2n_2 + n_3 + n_4); \tag{40}$$

then the coefficient turns out to be

$$\begin{aligned} &f_{b+c-(\xi_1-\frac{a}{2})}f_{b-(\xi_2-\frac{a}{2})} - f_{b-(\xi_1-\frac{a}{2})-1}f_{b+1+c-(\xi_2-\frac{a}{2})} \\ &= f_{n'-r}f_r - f_{r-n_1-1}f_{n'+n_1-r+1}. \end{aligned} \tag{41}$$

Next we compute the contributions from $\mathbf{J}$. We will need a result about representation of $\mathfrak{su}_2$ (see Harris–Fulton [FH] Exc 11.11):

$$S^m(\mathbb{C}^2) \otimes S^n(\mathbb{C}^2) = S^{m+n}(\mathbb{C}^2) + S^{m+n-2}(\mathbb{C}^2) + \cdots + S^{|m-n|}(\mathbb{C}^2). \tag{42}$$

Lemma 7.5.1: *The coefficient of $t_3^{\xi_3+2}t_4^{\xi_4+1}$ in $\mathbf{J}$ is the character of the representation $S_B^{n_3} \otimes S_C^{n_4}$ of $\mathfrak{su}_2(B) \oplus \mathfrak{su}_2(C)$.*

Proof: Define $\bar{t}_1 = t_1$, $\bar{t}_2 = t_2$, $\bar{t}_3 = t_3^{-1}$ and $\bar{t}_4 = t_4^{-1}$; then

$$\begin{aligned} \mathbf{J} &= t_5^{-1}\left\{\det(t_i^{j+\frac{a}{2}})_{i=3,\dots,6}^{j=0,\dots,3} + \det(\bar{t}_i^{j+\frac{a}{2}})_{i=3,\dots,6}^{j=0,\dots,3}\right\} \\ &\quad \times \Psi(t_3^{-1})\Psi(t_4^{-1})\left(1-\frac{t_5}{t_6}\right)^{-1}\left(1-\frac{1}{t_5t_6}\right)^{-1} \\ &= \left\{C^a\prod(t_i-t_j)(t_5-t_6) + C^{-a}\prod\left(t_i-\frac{1}{t_j}\right)\left(\frac{1}{t_5}-\frac{1}{t_6}\right)\right\} \\ &\quad \times t_5^{-1}(t_3t_4)^{\frac{a}{2}}(t_3-t_4)\Psi(t_3^{-1})\Psi(t_4^{-1})\left(1-\frac{t_5}{t_6}\right)^{-1}\frac{1}{1-C^{-2}} \\ &= \left\{C^{a+1}\prod\left(1-\frac{t_j}{t_i}\right)^{-1} + C^{-(a+1)}\prod\left(1-\frac{1}{t_it_j}\right)^{-1}\right\} \\ &\quad \times (t_3t_4)^{2+\frac{a}{2}}(t_3-t_4)\Psi(t_3^{-1})\Psi(t_4^{-1})\left(1-\frac{t_5}{t_6}\right)^{-1}\left(\frac{1}{C-C^{-1}}\right) \end{aligned} \tag{43}$$

where the products in the last two equations are taken over all $i = 3, 4$ and $j = 5, 6$.

Observe that

$$\begin{aligned} \left(1-\frac{t_5}{t_i}\right)^{-1}\left(1-\frac{t_6}{t_i}\right)^{-1} &= \sum_{n=0}^{\infty}\left(\frac{C}{t_i}\right)^n \sum_{p+q=n} B^{p-q} \\ &= \sum_{n=0}^{\infty}\left(\frac{C}{t_i}\right)^n \chi_B(n), \end{aligned} \tag{44}$$

$$\left(1-\frac{1}{t_i t_5}\right)^{-1}\left(1-\frac{1}{t_i t_6}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{1}{t_i C}\right)^n \sum_{p+q=n} B^{p-q}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{t_i C}\right)^n \chi_B(n). \tag{45}$$

Substitute (44) and (45) into (43) and we find that the coefficient of $t_3^{\xi_3+2} t_4^{\xi_4+1}$ is

$$\begin{aligned}&\left\{\chi_B\left(\frac{a}{2}-\xi_3\right)\chi_B\left(\frac{a}{2}-\xi_4\right)-\chi_B\left(\frac{a}{2}-\xi_3-1\right)\chi_B\left(\frac{a}{2}-\xi_4+1\right)\right\}\chi_C(\xi_3+\xi_4)\\ &=\chi_B(\xi_3-\xi_4)\chi_C(\xi_3+\xi_4)\quad \text{(by (42))}\\ &=\chi_B(n_3)\chi_C(n_4). \qquad \blacksquare\end{aligned}$$

7.6 We summarize what we have done so far:

Let $V(r,n')$ denote the representation of $\mathfrak{su}_2(B)\times\mathfrak{su}_2(C)$ that corresponds to the character in (41). *Let $\Omega = n_1\varpi_1 + n_2\varpi_2 + n_3\varpi_3 + n_4\varpi_4$ and*

$$W_\Omega = S_B^{n_3}\otimes S_C^{n_4}\otimes \sum_{a,b,c} S_A^c \otimes V\left(\frac{1}{2}a+b-\xi_2, N-\xi_1-\xi_2\right) \tag{46}$$

where the summation is taken over nonnegative integers a,b,c such that $c+2b+a=N$ and satisfy (34) *to* (35). *Then*

$$\mathrm{Res}^{E_7}_{\mathrm{SU}_2^3\times \mathrm{Spin}(8)}\pi(N\lambda) = \bigoplus_{\Omega=n_1\varpi+\cdots+n_4\varpi_4} W_\Omega\otimes\pi_{\mathfrak{d}_4}(\Omega) \tag{47}$$

where the sum is taken over $n_1+2n_2+n_3+n_4\le N$.

It remains to determine the summation in (46) which can be rewritten as

$$\sum_{c+2b+a=N} S_A^c\otimes V\left(\frac{a}{2}+b-\xi_2, N-\xi_1-\xi_2\right) = (n_2+1)\sum_r S_A^{n'-2r}\otimes V(r,n'), \tag{48}$$

where the conditions on a,b,c are translated into those in the summation for $n',r\in\mathbb{Z}$ (cf. (39) and (40)) which satisfies

$$\begin{aligned}0\le r\le \frac{n'+n_1}{2} &\quad \text{if } n'\ge n_1,\\ 0\le r\le n' &\quad \text{if } n'<n_1.\end{aligned}$$

Since the summation on the right hand side of (48) depends only on n' and n_1, we will denote the representation by $W(n',n_1)$. The next lemma will determine $W(n',n_1)$ and complete the proof of Proposition 7.1.1.

LEMMA 7.6.1:

$$\text{(a)}\qquad W(n,n_1) = S_A^{n_1}\otimes W(n,0),$$
$$\text{(b)}\quad W(n,0) = \mathrm{Sym}^n(S_A^1(\mathbb{C}^2)\otimes S_B^1(\mathbb{C}^2)\otimes S_C^1(\mathbb{C}^2)).$$

Proof: (a) By §2.3, $\mathfrak{e}_7$ has an automorphism s of order 3 which acts on $\mathfrak{d}_4$ as an outer automorphism. It stabilizes $\mathfrak{su}_2(A)\times\mathfrak{su}_2(B)\times\mathfrak{su}_2(C)$ by sending A to B, B to C, and C to A. The weight $(n+n_1)\lambda$ is invariant under this automorphism so the branching rule in (47) is symmetric with respect to s. Let $\Omega_i = n_1\varpi_1$ and $\Omega_j = n_1\varpi_3$; then $W_i = \sigma W_j$, that is,

$$S_{\tilde\alpha}^{n+n_1+8}\otimes W(n,n_1) = S_{\tilde\alpha}^{n+n_1+8}\otimes S_A^{n_1}\otimes W(n,0).$$

This completes the proof of part (a).
(b) The character of $W(n,0)$ is

$$\sum_{r=0}^{[n/2]}\chi_A(n-2r)(f_{n-r}f_r - f_{r-1}f_{n-r+1}) = \sum_{p+q=n}^{n} A^{p-q}f_pf_q. \tag{49}$$

Let the character of $\mathrm{Sym}^n(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$ be $\delta_n(A,B,C)$. Then

$$\prod_{i,j,k=\pm1}\left(1-xA^iB^jC^k\right)^{-1} = \sum_n \delta_n x^n. \tag{50}$$

Recall $t_5 = BC$ and $t_6 = C/B$ (cf. (21)) and hence

$$\begin{aligned}\prod_{i,j,k=\pm1}\left(1-xA^iB^jC^k\right)^{-1} &= \Psi(xA)\cdot\Psi\left(\frac{x}{A}\right)\\ &= \sum_n f_nA^nx^n\cdot\sum_n f_nA^{-n}x^n\\ &= \sum_n x^n\sum_{p+q=n}A^{a-b}f_pf_q.\end{aligned}$$

By comparing the coefficient of x^n, we complete the proof of part (b). ∎

8. Proof of Theorem 1.4.1

8.1 We refer to (5) and, by Proposition 7.1.1, $\Theta(\pi(\Omega))$ has K-types ($K = \mathrm{SU}_2\times\mathrm{SU}_2^3$)

$$(n_2+1)\bigoplus_N S_{\tilde\alpha}^{N+n_1+2n_2+n_3+n_4+8}\otimes\left(S_A^{n_1}\otimes S_B^{n_2}\otimes S_C^{n_3}\otimes\mathrm{Sym}^N(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)\right).$$

Let

$$H_\Omega = \mathbf{H}(\mathrm{Spin}(4,4), S_A^{n_1} \otimes S_B^{n_3} \otimes S_C^{n_4}[10 + n_1 + 2n_2 + n_3 + n_4]).$$

Comparing with Theorem 3.3.1(d), $\Theta(\pi(\Omega))$ has the same K-types as $n_2 + 1$ copies of H_Ω which is irreducible by §4.6. It remains to show that

$$\Theta(\Omega) = (n_2 + 1)H_\Omega.$$

By Corollary 3.7.2, H_Ω is the discrete series representation with infinitesimal character $\Omega+\overline{\rho}$. Theorem 4.4.1 and the information on the lowest K-types implies that

$$\Theta(\Omega) \supset (n_2 + 1)H_\Omega.$$

Since both representations have the same K-types, this is an equality and Theorem 1.4.1 is proven.

Alternatively, by Proposition 3.6.2, H_Ω satisfies Blattner's formula and the theorem follows immediately from Corollary 3.7.2.

9. Dual pair correspondences of $E_{7,4}$

9.1 Recall §2.4 that $E_{7,4}$ has maximal compact subgroup

$$K = \mathrm{SU}_2 \times_{\mu_2} \mathrm{Spin}(12)$$

and $V = V_M$ is the 32-dimensional half-spin representation of Spin(12) with highest weight $\lambda = \varpi_6$. The minimal representation π_Z of $E_{7,4}$ has K-types

$$\bigoplus_{N=0}^{\infty} S^{N+4}(\mathbb{C}^2) \otimes \pi(N\varpi_6). \tag{51}$$

9.2 First, we will identify the various subalgebras of $E_{7,4}$. Following Bourbaki [Bou1] Planche IV we denote the positive roots of the compact Lie subgroup $D_4 \subset E_{7,4}$ (resp. D_6) as

$$\varepsilon_i \pm \varepsilon_j, \quad 1 \leq i < j \leq 4 \text{ (resp. 6)}. \tag{52}$$

Therefore the $\frac{1}{2}$-spin representation of D_6 has highest weight

$$\varpi_6 = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_6).$$

Define

$$A = \exp\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big), \quad A' = \exp\Big(\frac{\varepsilon_1-\varepsilon_2}{2}\Big), \tag{53}$$

$$B = \exp\Big(\frac{\varepsilon_3+\varepsilon_4}{2}\Big), \quad B' = \exp\Big(\frac{\varepsilon_3-\varepsilon_4}{2}\Big), \tag{54}$$

$$C = \exp\Big(\frac{\varepsilon_5+\varepsilon_6}{2}\Big), \quad C' = \exp\Big(\frac{\varepsilon_5-\varepsilon_6}{2}\Big). \tag{55}$$

The SU_2's defined by A, B, C and A', B', C' agree with (4) and (3), respectively.

Suppose

$$\mathrm{Res}^{E_{7,4}}_{F_{4,4}\times \mathrm{SU}_2} \pi_Z = \bigoplus_n \Theta(n) \otimes S^n, \tag{56}$$

$$\mathrm{Res}^{E_{7,4}}_{\mathrm{Spin}(4,4)\times_{K_4} \mathrm{SU}_2^3} \pi_Z = \bigoplus_{a,b,c} \Theta(a,b,c) \otimes (S^a_{A'} \otimes S^b_{B'} \otimes S^c_{C'}), \tag{57}$$

where the sum is taken over all irreducible representations of SU_2 and SU_2^3 respectively. The objective of the rest of this paper is to compute the above dual pair correspondence given in Theorems 12.2.1 and 1.4.1.

10. Branching rules III

The main results of this section are 2 branching rules: Proposition 10.1.1 and Lemma 10.2.1. If the reader is willing to believe them, he may skip this section without loss of continuity.

10.1 Recall (52) and put $t_i = e^{\epsilon_i}$. We define a power series $\Psi(x)$ and its coefficients $f_n(t_3, t_4)$ for $n \in \mathbb{Z}$ by the following formula (cf. (32)):

$$\Psi(x) = \prod_{j=3,4} (1 - xt_j)^{-1}(1 - xt_j^{-1})^{-1} = \sum_{n=-\infty}^{\infty} f_n x^n.$$

Note that $f_n = 0$ for $n < 0$ and f_n is the character of $\mathrm{Sym}^n(\mathbb{C}^2_B \otimes \mathbb{C}^2_{B'})$ for $n \geq 0$ (cf. (54)). Also recall that $A = \sqrt{t_1 t_2}$ (cf. (53)).

PROPOSITION 10.1.1: *Let*

$$H = \mathrm{SU}_2(A) \times \mathrm{SU}_2(B) \times \mathrm{SU}_2(C) \times \mathrm{SU}_2(A') \times \mathrm{SU}_2(B') \times \mathrm{SU}_2(C')$$

and suppose

$$\mathrm{Res}^{\mathrm{Spin}(12)}_H \pi(N\varpi_6) = \sum_{a,b,c} W_N(a,b,c) \otimes (S^a_{A'} \otimes S^b_{B'} \otimes S^c_{C'}),$$

where $W_N(a,b,c)$ is a representation of $\mathrm{SU}_2(A) \times \mathrm{SU}_2(B) \times \mathrm{SU}_2(C)$. When $a \geq b+c$ and $a+b+c$ is an even integer, then

$$W_N(a,b,c) = S_B^c \otimes S_C^b \otimes \mathrm{Sym}^{N-a}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2).$$

By 12.1(a), the condition that $a+b+c$ is an even integer is superfluous. Proposition 10.1.1 is a consequence of the next lemma.

LEMMA 10.1.2: (a) *Suppose $a \geq b+c$ and $a+b+c$ is an even integer; then*

$$\begin{aligned} W_N(a,b,c) &= S_B^c \otimes W_{N-c}(a-c,b,0) = S_C^b \otimes W_{N-b}(a-b,0,c) \\ &= S_B^c \otimes S_C^b \otimes W_{N-b-c}(a-b-c,0,0); \end{aligned} \tag{58}$$

(b) $W_N(2a,0,0) = W_{N-2a}(0,0,0)$;
(c) $W_N(0,0,0) = \mathrm{Sym}^N(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$.

This section is devoted to proving Lemma 10.1.2. The proof in many areas resembles that of §7.

As a corollary of §7, we have the following branching rule:

$$\mathrm{Res}^{\mathrm{Spin}(12)}_{\mathrm{Spin}(8)\times \mathrm{SU}_2 \times \mathrm{SU}_2} \pi(N\varpi_6) = \sum_{c+2d+e=N} \pi(d\varpi_2 + c\varpi_3 + e\varpi_4) \otimes S_C^e \otimes S_{C'}^c. \tag{59}$$

Suppose

$$\begin{aligned} &\mathrm{Res}^{\mathrm{Spin}(8)}_{\mathrm{SU}_2(A')\times \mathrm{SU}_2(B')\times \mathrm{SU}_2(A)\times \mathrm{SU}_2(B)} \pi(d\varpi_2 + c\varpi_3 + e\varpi_4) \\ &\qquad = \sum_{a,b} (S_{A'}^a \otimes S_{B'}^b) \otimes V(a,b) \end{aligned} \tag{60}$$

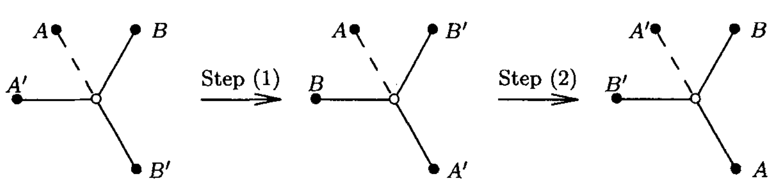
where $V(a,b)$ is a representation of $\mathrm{SU}_2(A) \times \mathrm{SU}_2(B)$. We will compute the character of $V(a,b)$ in the case $a \geq b+c$.

10.2 Recall that the Weyl group of D_4 is $\langle \pm 1 \rangle_{\Pi=1} \rtimes S_3$. We will apply the WCF to (60) and we have to be careful in choosing the right Weyl chamber of D_4:

(1) Acting by triality, we change the representation to $\pi(e\varpi_1 + d\varpi_2 + c\varpi_4)$.

(2) Next we apply the elements $(13)(24)$ and $\langle 1,-1,1,-1 \rangle$ in the Weyl group to the representation.

We illustrate the corresponding actions on the Dynkin diagrams.



In this case we have

$$A' = \epsilon_1 + \epsilon_2, \quad B' = \epsilon_1 - \epsilon_2, \quad A = \epsilon_3 - \epsilon_4, \quad B = \epsilon_3 + \epsilon_4.$$

Let

$$\Lambda = e\varpi_1 + d\varpi_2 + c\varpi_4. \tag{61}$$

Let W (resp. $\overline{W}$) be the Weyl group of D_4 (resp. D_2), $\chi(a,b)$ be the characters of $V(a,b)$, $\rho = (3,2,1,0)$ (resp. $\overline{\rho}$) be the half sum of the positive roots of D_4 (resp. D_2) and $t_i = e^{\epsilon_i}$. Applying the WCF to (60) and clearing denominators, we get

$$\begin{aligned} &\frac{\sum_{\sigma\in W} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho+\Lambda)}}{\prod'_{i,j}(1 - t_i^{-1}t_j^{\pm})} \\ &= e^{\rho-\overline{\rho}} \sum_{a,b} \sum_{\overline{\sigma}\in\overline{W}} \operatorname{sgn}(\overline{\sigma}) \exp(\overline{\sigma}(\frac{1}{2}(a+b+1), \frac{1}{2}(a-b))) \cdot \chi(a,b) \end{aligned} \tag{62}$$

where the product $\prod'_{i,j}$ is taken over

$$(i,j) \in (\{1,2\} \times \{3,4\}) \cup \{(3,4)\}.$$

The left hand side of (62) can be written as

$$\sum_{\sigma\in W} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho+\Lambda)} \prod_{i,j}{}' \left(\sum_{n=0}^{\infty} (t_i^{-1}t_j^{\pm})^n \right).$$

To find $\chi(a,b)$ for $a \geq b + c$, we only need to consider certain exponents in the numerator of (62) and then compute their coefficients. These exponents are of the form e^L where $L = (\epsilon_1, \dots, \epsilon_4)$ such that $\epsilon_1 > \epsilon_2 > \frac{1}{2}c + 1$. Since the Weyl group of D_4 acts transitively on its Weyl chambers, only the following sub sum on the left hand side of (62) will contribute:

$$\sum_{\sigma\in W'} \operatorname{sgn}(\sigma)\, e^{\sigma(\rho+\Lambda)} \cdot \prod_{i,j}{}' \left(\sum_{n=0}^{\infty} (t_i^{-1}t_j^{\pm})^n \right), \tag{63}$$

where $W' \subset W = \langle \pm 1 \rangle_{\Pi=1}^{4} \rtimes S_4$ is the subgroup

$$W' = S_2 \times \{\langle 1,1,-1,-1 \rangle \rtimes S_2\}. \tag{64}$$

We decompose (63) into 2 factors, each coming from a factor of W' in (64):

$$\text{Eq. } (63) = t_3 \cdot \mathbf{I} \cdot \mathbf{J} \tag{65}$$

where

$$\begin{aligned}
\mathbf{I} &= \left(t_1^{3+e+d+\frac{c}{2}}t_2^{2+d+\frac{c}{2}} - t_1^{2+d+\frac{c}{2}}t_2^{3+e+d+\frac{c}{2}}\right)\Psi(t_1^{-1})\Psi(t_2^{-1}),\\
\mathbf{J} &= \frac{1}{t_3}\sum_{\sigma\in\langle 1,1,-1,-1\rangle\rtimes S_2} \operatorname{sgn}(\sigma)\exp\left(\sigma\left(1+t_3^{\frac{c}{2}},t_4^{\frac{c}{2}}\right)\right)\\
&\quad\times(1-t_3^{-1}t_4)^{-1}(1-t_3^{-1}t_4^{-1})^{-1}\\
&= \chi_B(c).
\end{aligned}$$

Next we calculate the contributions from $\mathbf{I}$, that is, the coefficient of the term $t_1^{\frac{1}{2}(a+b)+3}t_2^{\frac{1}{2}(a-b)+2}$. This turns out to be

$$f_{e+d-\frac{b}{2}-(\frac{a}{2}-\frac{c}{2})}f_{d+\frac{b}{2}-(\frac{a}{2}-\frac{c}{2})} - f_{d-\frac{b}{2}-(\frac{a}{2}-\frac{c}{2})-1}f_{e+d+1+\frac{b}{2}-(\frac{a}{2}-\frac{c}{2})}. \tag{66}$$

Denote the above character by $F(e,d,b,a-c)$ so that the character of $W_N(a,b,c)$ is

$$\chi_B(c)\sum_{e+2d=N-c}\chi_C(e)F(e,d,b,a-c). \tag{67}$$

Proof of Lemma 10.1.2: (a) The sum in (67) depends only on $N-c$ and $a-c$. Therefore we conclude that

$$W_N(a,b,c) = S_B^c\otimes W_{N-c}(a-c,b,0).$$

Since the argument is symmetry with respect to B and C, we also get

$$W_N(a,b,c) = S_C^b\otimes W_{N-b}(a-b,0,c).$$

Applying the above twice, once each for B and C, we have (58). This proves part (a).

(b) and (c). By (67), the character of $W_N(2a,0,0)$ is

$$\begin{aligned}
&\sum_{e+2d=N}\chi_C(e)\left(f_{e+d-a}f_{d-a} - f_{d-a-1}f_{e+d+1-a}\right)\\
&= \sum_{a\le d\le N/2}\chi_C(N-2d)\left(f_{N-a-d}f_{d-a} - f_{d-a-1}f_{N-a+d+1}\right)\\
&= \sum_{d'=0}^{[N'/2]}\chi_C(N'-2d')(f_{N'-d'}f_{d'} - f_{d'-1}f_{N'-d'+1}) \qquad (68)
\end{aligned}$$

where $d'=d-a$ and $N'=N-2a$. By Lemma 7.6.1(b) and (49), (68) is the character of

$$\operatorname{Sym}^{N'}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2). \qquad ∎$$

LEMMA 10.2.1: *Let a, b, c be non-negative integers. Then the lowest K-type of the representation $\Theta(b+c, a+c, a+b)$ is*

$$S_{\tilde{\alpha}}^{4+a+b+c} \otimes S_A^a \otimes S_B^b \otimes S_C^c.$$

Proof: We apply the same analysis from (60) to (62). In order for the K-type to be lowest, it is necessary that (cf. (61))

$$\Lambda = c\varpi_1 + \frac{1}{2}(b+c)\varpi_4.$$

Next we argue as from (62) to (64) but we need to replace (64) with

$$W' = \{1, (23), (24)\} \times \{\langle 1, 1, -1, -1\rangle \rtimes S_2\}.$$

Let $\tau(a, b, c)$ be the lowest K-type; then its character can be computed in a similar manner as in (65) and (66):

$$\chi_C(c)\left(f_{\frac{a+b}{2}+2-\frac{b-a}{2}-2}\,\chi_B(a+b) - f_{\frac{a+b}{2}+1-\frac{b-a}{2}-2}\,\chi_B(a+b)\chi_C(1) + f_{\frac{a+b}{2}-\frac{b-a}{2}-2}\,\chi_B(a+b+2)\right).$$

The observation is that the formula within the bracket in the above equation is independent of the value of c. This allows us to conclude that

$$\tau(a, b, c) = S_C^c \otimes \tau(a, b, 0).$$

Since the representation τ is symmetric with respect to its entries a, b, c,

$$\tau(a, b, c) = S_A^a \otimes S_B^b \otimes S_C^c \otimes \tau(0, 0, 0).$$

The lemma follows as $\tau(0, 0, 0) = \mathbb{C}$. ∎

11. Branching rules IV

11.1 Consider $\mathrm{SL}_6 \subset \mathrm{GL}_6 \subset \mathrm{Spin}(12)$. Spin(12) and SL_6 have positive roots

$$\begin{aligned} \mathrm{Spin}(12) &: \epsilon_i \pm \epsilon_j \text{ where } 1 \le i < j \le 6, \\ \mathrm{SL}_6 &: \epsilon_i - \epsilon_j \text{ where } 1 \le i < j \le 6. \end{aligned}$$

We denote the half sum of the positive roots as

$$\begin{aligned} \rho &= \rho(D_6) = (5, 4, 3, 2, 1, 0), \\ \rho' &= \rho(A_5) = \left(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}\right). \end{aligned}$$

We denote the fundamental weights of GL_6 as

$$\begin{aligned} \varpi_1 &= \frac{1}{2}(1,-1,-1,-1,-1,-1), \\ \varpi_3 &= \frac{1}{2}(1,1,1,-1,-1,-1), \\ \varpi_5 &= \frac{1}{2}(1,1,1,1,1,-1), \end{aligned}$$

and the fundamental weight of Spin(12) as

$$\varpi_5 = \frac{1}{2}(1,1,1,1,1,-1).$$

11.2 We will prove a branching rule mentioned in Gross [G1] since we could not find a proof of it.

LEMMA 11.2.1:

$$\mathrm{Res}_{\mathrm{GL}_6}^{\mathrm{Spin}(12)} \pi(N\varpi_5) = \sum_{a+b+c=N} \pi(a\varpi_1 + b\varpi_3 + c\varpi_5).$$

Proof: The lemma is true for $N = 1$ (cf. [KP]). Since the representation $\pi(\varpi_5)$ of GL_6 appears in the branching rule for $N=1$, by Lemma 6.2.1, it suffices to prove the lemma for large N and c.

Suppose

$$\mathrm{Res}_{\mathrm{GL}_6}^{\mathrm{Spin}(12)} \pi(N\varpi_5) = \sum_i \pi(\lambda_i);$$

then applying the WCF and clearing denominators we get

$$\frac{\sum_{\sigma\in W(D_6)} \mathrm{sgn}(\sigma)\, e^{\sigma(\rho+N\varpi_5)}}{e^{\rho}\prod_{1\le i<j\le 6}(1-e^{-\epsilon_i-\epsilon_j})} = e^{-\rho'}\sum_i \sum_{\sigma\in S_6} \mathrm{sgn}(\sigma)\, e^{\sigma(\rho'+\lambda_i)}. \tag{69}$$

We only have to consider the coefficients of e^{λ} where $\lambda = \sum_i a_i\epsilon_i$ lies in the Weyl chamber of GL_6 (that is, $a_1 \ge \cdots \ge a_6$) in the power series expansion of the right hand side of (69) with $c = a_5 - a_6$ large. These exponents are contained in the sub sum

$$T = \frac{\sum_{\sigma\in H} \mathrm{sgn}(\sigma)\, e^{N\varpi_5+\sigma(\rho)}}{e^{\rho}\prod_{1\le i<j\le 6}(1-e^{-\epsilon_i-\epsilon_j})}$$

where $H \simeq S_6 \subset W(D_5)$ is the stabilizer of the fundamental weight ϖ_5. Recall formula (22),

$$\sum_{\sigma\in H} e^{\sigma(\rho)} = e^{\rho}\prod_{i,j}{}'(1-e^{-\epsilon_i+\epsilon_j})\prod_{i=1}^{6}(1-e^{-\epsilon_i-\epsilon_6})$$

where the product $\prod'_{i,j}$ is taken over all $1 \le i < j \le 5$. Substituting the above formula into T, we get

$$T = e^{N\varpi_5} \frac{\prod'_{i,j}(1 - e^{-\epsilon_i + \epsilon_j})}{\prod'_{i,j}(1 - e^{-\epsilon_i - \epsilon_j})}. \tag{70}$$

We have the partial fraction expansion

$$\begin{aligned} &\Big\{\prod_{i,j}{}'(1 - Xe^{-\epsilon_i - \epsilon_j})\Big\}^{-1} \\ &= \sum_{1 \le k < l \le 5} \Big\{\prod_{i,j}{}''(1 - e^{\epsilon_k + \epsilon_l - \epsilon_i - \epsilon_j}) \prod{}'(1 - Xe^{-\epsilon_k - \epsilon_l})\Big\}^{-1} \end{aligned} \tag{71}$$

where $\prod''_{i,j}$ is the sum over all $1 \le i < j \le 5$ such that i, j, k, l are distinct. Let $X = 1$ and put (71) into T so that T is a sum of terms of the form

$$Ae^{\omega} \left\{ \frac{\prod''_{i,j}(1 - e^{-\epsilon_i + \epsilon_j})}{\prod''_{i,j}(1 - e^{\epsilon_k + \epsilon_k - \epsilon_i - \epsilon_j})} \right\} \frac{1}{1 - e^{-\epsilon_i - \epsilon_j}} \tag{72}$$

for some integer A and some weight ω depending on $k < l$. Applying the partial fraction expansion to the factor in $\{\}$ in (72), we can write T as a sum of terms of the form

$$Ae^{\omega} \left\{(1 - e^{\epsilon_k + \epsilon_k - \epsilon_i - \epsilon_j})(1 - e^{-\epsilon_k - \epsilon_l})\right\}^{-1}.$$

Looking at the power series expansions of these terms, there is only one term such that it has summands e^{λ} where λ lies in the Weyl chamber of GL_6. This is the term where $i = 2$, $j = 3$, $k = 4$, $l = 5$, $\omega = N\varpi_5$ and $A = 1$. Its power series expansion is

$$\begin{aligned} &e^{N\varpi_5} \left(\sum_{m=0}^{\infty} e^{-m(\epsilon_4 + \epsilon_5)} \right) \left(\sum_{n=0}^{\infty} e^{-n(\epsilon_4 + \epsilon_5 - \epsilon_3 - \epsilon_4)} \right) \\ = \; &e^{N\varpi_5} \left(\sum_{m=0}^{\infty} e^{m(\varpi_3 - \varpi_5)} \right) \left(\sum_{n=0}^{\infty} e^{n(\varpi_1 - \varpi_3)} \right) \end{aligned}$$

and this proves the lemma. ∎

12. The main results for $E_{7,4}$

12.1 Let $\mathbb{C}^6$ denote the standard representation of $\mathrm{SU}_6 \supset \mathrm{Sp}_6$. Lemma 11.2.1 gives the Θ-correspondence for the dual pair $E_{6,4} \times U_1$ that was mentioned in Gross [G1].

THEOREM 12.1.1: *Suppose*

$$\mathrm{Res}^{E_{7,4}}_{E_{6,4}\times U_1} \pi_Z = \bigoplus_{n\in\mathbb{Z}} \Theta'(n) \otimes U_1^n.$$

Then

$$\Theta'(n) = \sigma(E_{6,4}, \mathrm{Sym}^n(\mathbb{C}^6)[6+n])$$

in the notation of §3.6. In particular, $\Theta'(0) = \pi_Y$ *of* $E_{6,4}$. ∎

THEOREM 12.2.1: *Suppose*

$$\mathrm{Res}^{E_{7,4}}_{F_{4,4}\times \mathrm{SU}_2} \pi_Z = \bigoplus_{n} \Theta(n) \otimes S^n \tag{73}$$

where the sum is taken over all irreducible representations of SU_2. *Then*

(a) $\Theta(2k+1) = 0$.

(b) $\Theta(2k) = \sigma(F_{4,4}, (\mathrm{Sym}^k \mathbb{C}^6)[6+k])$ *and it has infinitesimal character* $\rho(F_{4,4}) + k\varpi_1 - (3+\frac{1}{2}k)\tilde{\alpha}$ *where* ϖ_1 *is the highest weight of* $\mathbb{C}^6$ *(cf. [Bou1]).*

Proof: (a) The center -1 of SU_2 acts on $\Theta(n)\otimes S^n$ by multiplication by $(-1)^n$ but it acts trivially on π_Z.

(b) To save on notation let

$$\sigma_k = \sigma(F_{4,4}, \mathrm{Sym}^k \mathbb{C}^6[k+6]).$$

From the see-saw pair

$$\begin{array}{ccc} E_{6,4} & & \mathrm{SU}_2 \\ & \bowtie & \\ F_{4,4} & & U_1 \end{array}$$

and Theorem 12.1.1,

$$\mathrm{Res}^{E_{6,4}}_{F_{4,4}\times \mathrm{SU}_2} \pi_Y(E_{6,4}) = \bigoplus_{k=0}^{\infty} \Theta(2k). \tag{74}$$

By Proposition 4.2.1, (74) is contained in

$$\mathrm{Gr}(\mathbf{H}(E_{6,4}, \mathbb{C}[6])) = \bigoplus_{k=0}^{\infty} \mathbf{H}(F_{4,4}, \mathrm{Sym}^k \mathbb{C}^6[k+6])$$

and hence

$$\bigoplus_{k=0}^{\infty} \Theta(2k) \subset \bigoplus_{k=0}^{\infty} \sigma_k. \tag{75}$$

By Lemma 11.2.1 the lowest K-type of $\Theta(2k)$ occurs when $N = k$ and it equals $S_{\tilde{\alpha}}^{k+4} \otimes \mathrm{Sym}^k(\mathbb{C}^6)$ as representation of $K = \mathrm{SU}_2(\tilde{\alpha}) \times_{\mu_2} \mathrm{Sp}_6$ (cf. (51)). By Theorem 3.3.1(g), $\sigma_k \subset \Theta(2k)$ and, in view of (75), the proof is complete. ∎

12.2 Recall (57)

$$\mathrm{Res}^{E_{7,4}}_{\mathrm{Spin}(4,4)\times_{K_4}\mathrm{SU}_2^3} \pi_Z = \bigoplus_{a,b,c} \Theta(a,b,c) \otimes (S^a_{A'} \otimes S^b_{B'} \otimes S^c_{C'}). \tag{76}$$

We would like to make the following remarks:

(a) $\Theta(a,b,c)$ is trivial unless $a+b+c$ is an even integer. This follows from the fact that $x = \langle -1,-1,-1\rangle$ lying in the center of SU_2^3 acts on

$$\Theta(a,b,c) \otimes (S^a_{A'} \otimes S^b_{B'} \otimes S^c_{C'})$$

by multiplication by $(-1)^{a+b+c}$ but x acts trivially on π_Z.

(b) The S_3 outer automorphism of $\mathrm{Spin}(4,4) \times_{K_4} \mathrm{SU}_2^3$ permutes the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$ (cf. §2.3). Since the S_3 action comes from inner automorphisms of $E_{7,4}$, the above correspondence in (76) respects this action. Therefore, without loss of generality, we may assume that $a \geq b, c$.

(c) From (a), since $a+b+c$ is even, there exist integers a', b', c' such that

$$a = b' + c', \quad b = c' + a', \quad c = a' + b';$$

a, b, c will form the sides of a triangle if and only if a', b', c' are non-negative. If a', b', c' are strictly positive, we will call this case **I**. Otherwise it will be called case **II**.

12.3 Let $S(a,b,c) = S^a_A \otimes S^b_B \otimes S^c_C$ be a representation of $M = \mathrm{SU}_2^3 \subset \mathrm{Spin}(4,4)$ (cf. Proposition 4.6.1).

THEOREM 12.4.1: (a) $\Theta(a,b,c) = 0$ *if* $a+b+c$ *is not an even integer.*

(b) *Case* **I***: Let* a, b, c *be 3 positive integers. Then*

$$\Theta(b+c, a+c, a+b) = \sigma(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c])$$

and it satisfies the following exact sequence:

$$0 \to \Theta(b+c, a+c, a+b) \to \mathbf{H}(\mathrm{Spin}(4,4), S(a,b,c)[6+a+b+c]) \to$$
$$\to \mathbf{H}(\mathrm{Spin}(4,4), S(a-1,b-1,c-1)[7+a+b+c]) \to 0.$$

It has infinitesimal character

$$\left(a+1+\frac{b+c}{2}, 1+\frac{b+c}{2}, \frac{b+c}{2}, \frac{b-c}{2}\right).$$

(c) *Case* **II**: *Suppose $a \geq b+c$, $a+b+c$ is an even integer. Then*

$$\Theta(a,b,c) = \mathbf{H}(\mathrm{Spin}(4,4), S(0,c,b)[6+a]) \tag{77}$$

and it has infinitesimal character

$$\left(\frac{a}{2}+1, \frac{a}{2}, \frac{b+c}{2}+1, \frac{b-c}{2}\right).$$

If $a = b+c+2$, $\Theta(a,b,c)$ is the limit of the discrete series representation. If $a \geq b+c+4$, $\Theta(a,b,c)$ is the discrete series representation.

Proof: (b) Let a, b, c be non-negative integers. By Lemma 10.2.1, we get

$$\Theta(b+c, c+a, a+b) \supset \sigma(S(a,b,c)[6+a+b+c]). \tag{78}$$

In view of the see-saw pair in (8), the SU_2-invariant subspace in $S(a,b,c)$ is either 0 or 1 dimensional. The latter will occur if and only if a, b, c form the sides of a triangle (cf. §12.3(c)). Therefore by Theorem 12.2.1

$$\mathrm{Res}^{F_{4,4}}_{\mathrm{Spin}(4,4)} \pi_X(F_{4,4}) = \bigoplus_{a,b,c} \Theta(b+c, c+a, a+b).$$

(16) and (17) of Proposition 4.6.1 show that (78) is indeed an equality and this proves part (a).

(c) The case $a = b+c$ has been proven in (b). Assume $a \geq b+c+2$ and, by (15),

$$\mathbf{H}(\mathrm{Spin}(4,4), S(0,c,b)[6+a]) \tag{79}$$

is irreducible and unitarizable. When $a \geq b+c+4$ it is the discrete series representation (cf. §4.6). Otherwise it is the limit of the discrete series. Hence it remains to prove (77). By Lemma 10.1.1, the K-type of $\Theta(a,b,c)$ is

$$\bigoplus_{n=0}^{\infty} S_{\tilde{\alpha}}^{4+a+n} \otimes \mathrm{Sym}^n(\mathbb{C}_A^2 \otimes \mathbb{C}_B^2 \otimes \mathbb{C}_C^2) \otimes S(0,c,b).$$

This equals the K-types of (79) by Theorem 3.3.1(d) so the theorem follows from Theorem 4.4.1. ∎

References

[Bo] H. Boerner, *Representations of Groups*, North-Holland, Amsterdam–London, 1970.

[Bou1] N. Bourbaki, *Éléments de Mathématique: Groupes et Algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[Bou2] N. Bourbaki, *Éléments de Mathématique: Groupes et Algèbres de Lie*, Chapitres 7 et 8, Hermann, Paris, 1975.

[Ca] E. Cartan, *Oeuvres Complètes*, Partie I. (French) *Groups de Lie*, Second edition, CNRS, Paris, 1984.

[FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Text in Mathematics 129, Springer-Verlag, Berlin, 1991.

[G1] B. Gross, *Letter to Kostant.*

[GS] B. Gross and G. Savin, *Motives with Galois group of type G_2: An exceptional theta-correspondence*, Compositio Mathematica **114** (1998), 153–217.

[GW1] B. Gross and N. Wallach, *A distinguished family of unitary representations for the exceptional groups of real rank = 4*, in *Lie Theory and Geometry: In Honor of Bertram Kostant*, Progress in Mathematics, Vol. 123, Birkhäuser, Boston, 1994.

[GW2] B. Gross and N. Wallach, *On quaternionic discrete series representations and their continuations*, Journal für die Reine und Angewandte Mathematik **481** (1996), 73–123.

[Ho] R. Howe, *Remarks on classical invariant theory*, Transactions of the American Mathematical Society **313** (1989), 539–670.

[HPS] J. S. Huang, P. Pandzic and G. Savin, *New dual pair correspondences*, Duke Mathematical Journal **82** (1996), 447–471.

[HS] H. Hecht and W. Schmid, *A proof of Blattner's conjecture*, Inventiones Mathematicae **31** (1975), 129–154.

[JV] H. P. Jakobsen and M. Vergne, *Restrictions and expansions of holomorphic representations*, Journal of Functional Analysis **34** (1979), 29–53.

[KP] W. G. McKay and J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics, Volume 69, M. Dekker, New York, 1981.

[KV] A. Knapp and D. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University Press, Princeton, NJ, 1995.

[Li1] J.-S. Li, *Two reductive dual pairs in groups of type E*, Manuscripta Mathematica **91** (1996), 163–177.

[Li2] J.-S. Li, *On the discrete spectrum of $(G_2, PGSp_6)$*, Inventiones Mathematicae **30** (1997), 189–207.

[Li3] J.-S. Li, *The correspondences of infinitesimal characters for reductive dual pairs in simple Lie groups*, preprint.

[L1] H. Y. Loke, *Lie algebras with S_4 structures*, preprint.

[L2] H. Y. Loke, *Exceptional Lie Algebras and Lie Groups*, Part 2, Harvard Thesis, 1997.

[Ma] S. Martens, *The characters of the holomorphic discrete series*, Proceedings of the National Academy of Sciences of the United States of America **72** (1975), 3275–3276.

[Pr] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloquium Mathematicum **70** (1996), 93–102.

[S1] W. Schmid, *Homogeneous complex manifolds and representations of simple Lie groups*, Dissertation, University of California, Berkeley, 1967; reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Mathematical Surveys and Monographs, Vol. 31, American Mathematical Society, Providence, 1989, pp. 223–286.

[S2] W. Schmid, *Some properties of square-integrable representations of semi-simple Lie groups*, Annals of Mathematics **102** (1975), 535–564.

[SW] W. Schmid and J. Wolf, *A vanishing theorem for open orbits on flag manifolds*, Proceedings of the American Mathematical Society **92** (1984), 461–464.

[V] D. Vogan, Jr, *Unitary Representations of Reductive Groups*, Annals of Mathematics Studies 118, Princeton University Press, 1987.

[W1] H. W. Wong, *Dolbeault cohomology realization of Zuckerman modules associated with finite rank representations*, Dissertation, Harvard University, 1991.

[W2] H. W. Wong, *Dolbeault cohomology realization of Zuckerman modules associated with finite rank representations*, Journal of Functional Analysis **129** (1995), 428–454.